A behavioral-geometric viewpoint to nonlinear coprime factorization for dissipative systems

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Abstract—In this paper we deal with a nonlinear extension to the classical theory of linear balancing based on coprime factorizations. In particular, we present our viewpoint on nonlinear coprime factorizations as it can be characterized from our differential geometric balancing framework based on Hilbert manifolds presented in previous papers. The approach presented in this paper distinguishes from other nonlinear extensions, published earlier by several other authors, by the extensive use of differentiable Lie semigroups of diffeomorphisms.

Index Terms—Nonlinear systems, model approximation, model reduction, differential geometric methods, geometric approaches, factorization methods.

I. INTRODUCTION

The classical theory of coprime stable factorizations is fundamental in $\mathcal{H}_\infty$-robust control. When a stable transfer matrix $P(s) \in \mathcal{RH}_\infty$ is right factorized $P(s) \triangleq N(s)D^{-1}(s)$ or left factorized $P(s) \triangleq D^{-1}(s)\tilde{N}(s)$ by the stable transfer matrices $N(s), \tilde{N}(s) \in \mathcal{RH}_\infty$ and (theinvertible) $D(s), \tilde{D}(s) \in \mathcal{RH}_\infty$, there may exist pole-zero cancellations in such factorizations unless their factors $N(s)$ and $D(s)$ are coprime. We say that $N(s)$ and $D(s)$ are right coprime if the Bezout equation $X_r(s)N(s) + Y_r(s)D(s) = I$ is satisfied and they are left coprime if the Bezout equation $N(s)X_l(s) + \tilde{D}(s)\tilde{Y}_l(s) = I$ is satisfied. A double coprime factorization occurs when there are factor matrices $N(s), \tilde{N}(s), D(s), \tilde{D}(s) \in \mathcal{RH}_\infty$ such that $P(s) = N(s)D^{-1}(s) = D^{-1}(s)\tilde{N}(s)$ and both Bezout equations are satisfied for some $X_r(s), Y_r(s), X_l(s), Y_l(s)$, see e.g. (Vidyasagar, 1985; Obinata and Anderson, 2001). Important concepts in robust control like internal stability, the Youla-Kučera parametrization, etc., were consequently developed using these factorizations. In particular, extensions for linear state-space realizations and their relationships with the balancing theory are well established. While balancing of the normalized left coprime factorization is discussed in (Ober and McFarlane, 1989), balancing of the normalized right coprime factorization is discussed in (Meyer, 1990). Balancing of state-space doubly normalized coprime representations are discussed in (Nett et al., 1984). The approach of LQG balancing by (Opdenacker and Jonckheere, 1985) was generalized in the linear behavioral approach by (Weiland, 1991).

In this paper we deal with a nonlinear extension of coprime stable factorizations. There is a huge list of references on this topic, see e.g. (van der Schaft, 2000; Helton and James, 1999) for an appropriate account of results. Earlier works on this subject relevant to this paper are annotated in (Scherpen and van der Schaft, 1994; Scherpen, 1994) where the authors contributed with a nonlinear extension of the concepts of (left and right) normalized coprime factorizations using the concepts of inner and co-inner nonlinear systems. Their balancing method is posed in terms of Hamilton-Jacobi-Bellman Equations.

Just like in the linear robust control theory, nonlinear coprime factorizations are useful for nonlinear controller reduction, see (Pavel and Fairman, 1997). Moreover, a generalization of the linear Youla-Kučera parametrization, provides all the stabilizing controllers of a nonlinear system using stable kernel representations, see (van der Schaft, 2000), and references therein.

With the recent developments of our differential geometric framework for nonlinear behavioral balanced reduction, e.g. (Lopezlena, 2004; Lopezlena and Scherpen, 2006; Lopezlena, 2006b; Lopezlena, 2006a; Lopezlena, 2008a; Lopezlena, 2008b), we have been able to extend to nonlinear dissipative systems alternative interpretations for Schmidt decomposition, past and future Gramians, eigenfunction decomposition, principal frame-balancing conditions, etc. Therefore, it is natural to inquire about the extension of the classical theory of stable coprime factorizations under this framework, being precisely the purpose of this paper to provide the first answers to such questions.

The relevance of the results presented in this work are associated to the role that stable factorizations have played in the theoretical development of model reduction algorithms for nonlinear control systems, along with the Author’s interest in developing numerical methods for plant and controller reduction.

The paper structure follows: In Sec. II we recall geometric antecedents about our behavioral framework. In Sec. III we view the behavioral operator as a paradigm for characterization of isometric operators. In Sec. IV, we show our main results about properties of adjoint systems and factorization.
II. Geometric Antecedents

In (Lopezlena, 2006b; Lopezlena, 2006a) the author introduced a geometric framework characterizing the trajectories and invariant functions of two intertwined systems supported by two Hilbert manifolds dualized by a duality pairing. The direction of evolution is ruled by the sign of the time rate, defining a strictly increasing (forward-time) or strictly decreasing (backward-time) evolution. Denote the time direction in the conventional forward-time evolution by $t = \{t | t \in \mathbb{R}^+\}$ and a backward-time evolution by $\tau = \{\tau | \tau = -t, t \in t\}$. Furthermore an initial time $t_0 \in t$ partitions the time interval in past and future semi-intervals, introducing two half-spaces of (past and future) semi-trajectories.

While the future behavior is supported on $\mathcal{B}^+ \subset t \times \mathcal{W} \triangleq \mathbb{R}^n$, there is a dual past behavior on $\mathcal{B}^- \subset \tau \times \mathcal{W}^* \triangleq \mathbb{R}^{-n}$. Thus, although the behavior is uniquely defined by $\mathcal{B} \subset \mathbb{T} \times \mathcal{W}$, $\mathcal{B}$ is partitioned into two parts $\mathcal{B} = \mathcal{B}^- \oplus \mathcal{B}^+$ where $\mathcal{B}^- \subset \mathcal{W} \times \mathcal{B}$ and $\mathcal{B}^+ \subset \mathcal{W} \times \mathcal{B}$ are two dual half-spaces with duality identified at their boundary $\mathcal{B}_0$ (the initial or present conditions). In consequence, every trajectory $w(t) \in \mathcal{B} \subset \mathcal{W} \times \mathcal{B}$ admits a partition in a positive semi-trajectory $w(t) \in \mathcal{B} \times \mathcal{B}^+$, $t^+ \in t$ and a negative semi-trajectory $w(t) \in \mathcal{B}^+ \times \mathcal{B}^+$, $t^- \in \tau$.

Consider the $\Sigma$-class of continuous-time nonlinear systems defined by

$$
\Sigma^+ : \begin{align*}
\dot{x}(t) &= f(x(t)) + g(x(t))u(t), \\
y(t) &= h(x(t)),
\end{align*}
$$

(1)

$x \in \mathbb{R}^n$ are local coordinates for a $C^\infty$ state space manifold $\mathcal{M}$, $f$, $g$ and $h$ are $C^\infty$. The set of external variables $\mathcal{W} \approx \mathbb{R}^p$, $p + q \leq \omega$, includes $u \in \mathcal{U} \subset \mathbb{R}^p$ and $y \in \mathcal{Y} \subset \mathbb{R}^q$ as subsets. We assume that $f$ and $h$ are Lipschitz continuous in $x$ and additionally $x$ and $y$ are locally square integrable.

Associated to the (smooth) time varying vector field $x \mapsto F(x, u(t))$, $F(x, u(t)) \triangleq f(x(t)) + g(x(t))u(t)$, where the (piecewise constant) control input $u(t)$ is a map $u : t \mapsto \mathcal{U}$, there is a family of vector fields denoted by $\mathcal{F}_u = \{F_u : u \in \mathcal{U}\}$. The (phase) trajectories of this system are continuous curves $q(t)$ in $\mathcal{M}$ that define on an interval $[0, T]$, integral curves of the family $\mathcal{F}_u$ if there exists a partition $0 = t_0 \leq t_1 \leq \cdots \leq t_m = T$ and associated vector fields $\xi_1, \ldots, \xi_m \in \mathcal{F}_u$ such that the restriction of $q(t)$ to each open interval $(t_i, t_{i+1})$, $i = 0, \ldots, m$, is differentiable and such that $d\xi(t)/dt = \xi_i(q(t))$, $\phi(0) = \phi^0$, $t \in (t_i, t_{i+1})$, $i = 0, \ldots, m$.

For the evolution of system $\Sigma$ backwards in time, the dynamical relation in (1) is expressed as

$$
\Sigma^- : \begin{align*}
\dot{x}(\tau) &= -f(\hat{x}(\tau)) - g(\hat{x}(\tau))\hat{u}(\tau), \\
\hat{y}(\tau) &= h(\hat{x}(\tau)),
\end{align*}
$$

(2)

where we assume $\hat{x}$ (the costate) are local coordinates for a $C^\infty$ dual manifold $\mathcal{M}^*$, $f$, $g$ and $h$ are $C^\infty$. The set of dual external variables $\mathcal{W}^*$ includes $\hat{u} \in \mathcal{U}^*$ and $\hat{y} \in \mathcal{Y}^*$ as subsets. Here is also a (smooth) time varying vector field $\hat{x} \mapsto -F(\hat{x}, \hat{u}(\tau))$, $F(\hat{x}, \hat{u}(\tau)) = f(\hat{x}(\tau)) + g(\hat{x}(\tau))\hat{u}(\tau)$ where the (piecewise constant) control input $\hat{u}(\tau)$ is a map $\hat{u} : \tau \mapsto \mathcal{U}^*$ and there is a family of vector fields $\mathcal{F}_u = \{F_u : \hat{u} \in \mathcal{U}^*\}$.

Example 2.1: The positive semi-trajectories of system

$$
\Sigma^+ : \begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t),
\end{align*}
$$

(3)

(using standard notation, the $\Sigma$-class of continuous-time linear time-invariant systems) with (state) space $\mathcal{M} = \mathbb{R}^n$ evolve in $w(t) \in \mathcal{W} \times \mathcal{B}$ and the negative semi-trajectories of their backward-time system (with $\tau := -t$)

$$
\Sigma^- : \begin{align*}
\dot{x}(\tau) &= -Ax(\tau) + Bu(\tau), \\
\hat{y}(\tau) &= C\hat{x}(\tau),
\end{align*}
$$

(4)

with (costate) space $\mathcal{M}^* = \mathbb{R}^n$ evolve in $\hat{w}(\tau) \in \mathcal{W} \times \mathcal{B}^*$, $\tau^+ \in \tau$.

For the description of dynamical systems, our approach relies on differentiable Lie semigroups of diffeomorphisms. Therefore we recall their definition in this section in order to characterize evolutionary operators. For an introduction on differentiable (Lie) semi-groups of diffeomorphisms, see e.g. (Clément and Heijmans, 1987; Hilgert and Neeb, 1993; Knapp, 2002) and for a differential-geometric treatment see (Graham, 1983).

Before presenting a definition of dynamical systems useful for our purposes, we need the following.

Definition 2.1: A family $\{\Phi(x, t), t \in t, x \in x \subset M \subset C\}$ in a class of bounded operators in $\mathcal{M}$ is called a 1-parameter semigroup if it is such that the mapping $\Phi : \mathbb{R}^1 \times D \rightarrow D$, $\Phi(t, x) = \phi^t(x)$ depends smoothly on $t \in \mathbb{R}^+$; $\phi^0(x) = x$ and $\phi^{t_2} \circ \phi^{t_1}(x) = \phi^{t_2 + t_1}(x)$. Such semigroup is called strongly continuous (or $C_0$-semigroup) if $t \mapsto \phi^t(x)$ is continuous on $[0, \infty)$ for every $x \in \mathcal{M}$ (Clément and Heijmans, 1987). If a semigroup carries the structure of an $n$-dimensional smooth manifold $\mathcal{M}$ satisfying the semigroup property, it is called a Lie-semigroup. A Lie semi-group is called differentiable if each operation $\circ : S_G \times S_G \rightarrow S_G$ yields a differentiable map (Graham, 1983). If a Lie-semigroup satisfies the inversion operation $i : G \rightarrow G$, with smooth maps between manifolds, $i(G) = g^{-1}$, $g \in G$ then $G$ is called a Lie-group.

In this paper we deal with bounded, differentiable Lie $C_0$- (semi-) groups. When the semigroup represents a dynamical system, we would prefer to call it evolutionary operator:

Definition 2.2 (Dynamical system): A dynamical system is the triad $(t, \mathcal{M}, \Phi^t)$ in forward-time or the triad $(\tau, \mathcal{M}^*, \Theta^\tau)$ in backward-time where $t$ (resp. $\tau$) is a semi-interval of evolution, $\mathcal{M}$ (resp. $\mathcal{M}^*$) defines the state-space (resp. the costate-space) and the semigroup $\Phi^t$ (resp. $\Theta^\tau$) is called an evolutionary operator.

The theory of stability of these evolutionary operators is well established, see e.g. (Abraham and Marsden, 1978; Bhatia and Szego, 2002). The following concept of isometry serves us to provide appropriate definitions for duality pair-
ings, adjoint and self-adjoint differential operators on Hilbert manifolds, see (Lopezena, 2006a) for further details:

Definition 2.3 (Local isometry): Let \((W, (\cdot, \cdot)_W)\) and \((V, (\cdot, \cdot)_V)\) be two Riemannian manifolds. An injective isometric operator is a metric-preserving map \(\Lambda : W \rightarrow V\) (with tangent map \(\Lambda_* : TW \rightarrow TV\)) s.t. \(<\xi, \zeta>_W = <\Lambda_*\xi, \Lambda_*\zeta>_V\) for \(\xi, \zeta \in TW\). Inversely, an injective co-isometric operator is a metric-preserving map \(\Lambda : V \rightarrow W\) s.t. \(<\Lambda_*\alpha, \Lambda_*\beta>_W = <\alpha, \beta>_V\) for \(\alpha, \beta \in TV\). If both maps are bijective, s.t. \(\Lambda^\dagger = \Lambda^{-1}\), it is simply called a local isometry.

Since lengths of curves, areas of regions and angles between curves remain undistorted by them, local isometries are structure preserving. Based on this consider the following

Definition 2.4 (Adjoint and self-adjoint operator): Let \((W, (\cdot, \cdot)_W)\) and \((V, (\cdot, \cdot)_V)\) be two dual Riemannian manifolds. A differential operator \(\Lambda : W \rightarrow V\) (with tangent map \(\Lambda_* : TW \rightarrow TV\)) is said to have an adjoint differential operator \(\Lambda^+: W^* \rightarrow V\) if it satisfies

\[
<\Lambda^\dagger\alpha, \xi>_W = <\alpha, \Lambda\xi>_V, \quad \xi \in TW, \alpha \in T^*W. \tag{5}
\]

A differential operator \(\Omega : W \rightarrow W\) is called self-adjoint if it satisfies \((\Omega\xi, \zeta)_W = (\Omega\zeta, \xi)_W\) for \(\xi, \zeta \in TW\).

Remark 2.1: Isometric operators on Hilbert spaces are commonly used in linear control theory, see (Feintuch, 1998). Let \(G(s) \in \mathcal{RH}_\infty\) be an all pass transfer function, i.e. \(||G(s)||_\infty = 1\), then its associated operator \(M_G\) is isometric, i.e. \((\hat{x}, \hat{x})_M = (M_G\hat{x}, M_G\hat{x})_M\) (Green and Limebeer, 1995). When the transfer function \(G(s)\) (with co-adjoint \(G(-s)^* \in \mathcal{RH}_\infty\)) satisfies the frequency domain condition \(G(-s)G(s) = I\), \(\forall s \in C\), \(G(s)\) is called inner. When \(G(s)G(-s)^* = I, \forall s \in C\), \(G(s)\) is called co-inner. The nonlinear extensions to inner and co-inner systems were defined in (Scherpen and van der Schaft, 1994). Nevertheless since later in Prop. 4.2 we show that the associated evolutionary operators to such systems are isometric operators on Hilbert manifolds, in this paper we prefer to use Def. 2.4 for adjoint and self adjoint operators on Hilbert manifolds instead.

Consider the past behavioral trajectories by \(W^- = \mathcal{U} \oplus V^*\) supported on the Riemannian manifold \((\mathcal{B}^-, (\cdot, \cdot)_{\mathcal{B}^-})\) and the future behavioral trajectories by \(W^+ = \mathcal{U} \oplus \Omega\) on the Riemannian manifold \((\mathcal{B}^+, (\cdot, \cdot)_{\mathcal{B}^+})\).

Assumption 2.1: Throughout this paper we assume the halfspaces \((\mathcal{B}^-, (\cdot, \cdot)_{\mathcal{B}^-})\) and \((\mathcal{B}^+, (\cdot, \cdot)_{\mathcal{B}^+})\), carry the structure of Riemannian Hilbert manifolds, i.e., differentiable manifolds locally modelled on separable Hilbert spaces whose inner products (in the contractions \(i(\cdot\), hats denote duality)

\[
<\alpha^1, \alpha^2>_{\mathcal{B}^-} \overset{df}{=} \int_0^\infty i_{\xi_1^1} M_1 \alpha^2 d\mu(t), \tag{6}
\]

\[
<\xi^1, \xi^2>_{\mathcal{B}^+} \overset{df}{=} \int_0^\infty i_{\xi_1^2} M_2 \xi^2 d\mu(t), \tag{7}
\]

for \(\alpha^1, \alpha^2 \in \mathcal{T}_{w^-} \mathcal{B}^-, \xi^1, \xi^2 \in \mathcal{T}_{w^+} \mathcal{B}^+\) define metrics \(\mathcal{B}^-_g, \mathcal{B}^+_g\) for \(T_w \mathcal{B}^-\) and \(T_w \mathcal{B}^+, \forall w(t) \in \mathcal{B}\).

Duality of the Hilbert manifolds \((\mathcal{B}^-, (\cdot, \cdot)_{\mathcal{B}^-})\) and \((\mathcal{B}^+, (\cdot, \cdot)_{\mathcal{B}^+})\) is identified with the duality pairing

\[
<\xi^+, \alpha^->_{\mathcal{B}^- \times \mathcal{B}^+} \overset{df}{=} \int_0^\infty i_{\xi^+} \tilde{\Gamma}^\dagger \alpha^- d\mu(t), \tag{8}
\]

for \(\xi^+ \in T\mathcal{B}^+, \alpha^- \in T\mathcal{B}^-\) and an isometry \(\tilde{\Gamma} : \mathcal{B}^- \rightarrow \mathcal{B}^+\) (the behavioral operator) s.t. \(\|\alpha^-\|_{\mathcal{B}^-} = ||\xi^+||_{\mathcal{B}^+}\).

The behavioral operator is a nonlinear generalization of the linear Hankel operator and it plays the same role the Hankel operator has played in linear balancing theory.

Remark 2.2: The behavioral operator \(\tilde{\Gamma}\) is a bijective isometry i.e. there is a co-isometry \(\tilde{\Gamma}^\dagger : \mathcal{B}^\dagger \rightarrow \mathcal{B}^-\) s.t. \(\|\alpha^-\|_{\mathcal{B}^-} = ||\xi^+||_{\mathcal{B}^+}\) for an equivalent duality pairing

\[
<\xi^+, \alpha^->_{\mathcal{B}^+ \times \mathcal{B}^-} \overset{df}{=} \int_0^\infty i_{\xi^+} \tilde{\Gamma}^\dagger \alpha^- d\mu(t). \tag{9}
\]

We assume throughout that all the evolutionary operators representing a dynamical system are supported by such Hilbert manifold structures and from hereunder we will use them at will.

The properties of the behavioral operator have been characterized elsewhere, see e.g. (Lopezena, 2006a), nevertheless, since this operator serves as a paradigm to characterize all the system factorizations of this paper, in the following section further aspects of the behavioral operator are discussed. Influenced by the past \(\mathcal{B}^- \subset \mathcal{W}^* \times \mathcal{t}\), the system trajectories are used to define the future behavior \(\mathcal{B}^\dagger \subset \mathcal{W} \times \mathcal{t}\). The behavior and the system trajectories are related by two storage functions associated to system \(\Sigma\), the backward-time required supply,

\[
S^r_\Sigma (x_0, r_r) = - \sup_{\tilde{\omega}(\cdot) \in \mathcal{U}^*} \int_0^\infty r_r(w^r(\tau)) d\tau, \tag{10}
\]

and the available storage,

\[
S_a (x_0, a_r) = \sup_{w(\cdot) \in \mathcal{W}} \int_0^\infty a_r(w_f(\tau)) d\tau, \tag{11}
\]

where the function of external signals defined by \(r : \mathcal{W} \rightarrow \mathcal{R}^1, r(w(t))\), is called supply rate relative to \(S^r_\Sigma\) or \(S_a\) respectively.

III. THE BEHAVIORAL OPERATOR AS A PARADIGM

An assumption made throughout the paper follows:

Assumption 3.1: The dualized Hilbert manifolds \((\mathcal{B}^-, (\cdot, \cdot)_{\mathcal{B}^-})\) and \((\mathcal{B}^+, (\cdot, \cdot)_{\mathcal{B}^+})\) have past and future induced metrics given by

\[
S^r_\Sigma (\hat{w}_0, r_r) \overset{df}{=} \mathcal{B}^-_g (\hat{w}_-, 0) = <\alpha_-, \alpha^->_\mathcal{B}^-, \tag{12}
\]

\[
S_a (w_0, a_r) \overset{df}{=} \mathcal{B}^+_g (w^+, 0) = <\xi^+, \xi^+_r>_\mathcal{B}^+. \tag{13}
\]

were \(\mathcal{B}^-_g (\hat{w}_0, r_r)\) is the required supply and \(S_a (w_0, r_a)\) is the available storage.

For instance, a dissipative system \(\Sigma\) supported by the Hilbert manifold structure in Assum. 3.1, may have arbitrary trajectories such that the metric of the past trajectory results
in the required supply $S_a^+$ and the metric of the future trajectory equals the $S_a$ available storage. Such arbitrary trajectories satisfy the dissipation inequality, such that $0 \leq S_a \leq S_r$, nevertheless there is a set of invariant trajectories such that $S_a$ is related to $S_r$ by stationary values (an invariant function). These invariant trajectories can be determined as the invariants of the behavioral operator, introduced in (Lopezlena and Scherpen, 2006a).

**Proposition 3.1:** Consider the system $\Sigma$ supported by the past and future Hilbert manifolds in Assum. 2.1 and 3.1, where $\langle u(t), y(t) \rangle \equiv w(t) \in \mathcal{B}$. The (behavioral operator) map $\bar{\Gamma} : L_2[-T, 0] \rightarrow L_2(0, T)$, $\bar{\Gamma} : \mathcal{B}^- \rightarrow \mathcal{B}^+$ defined by

$$
\begin{bmatrix}
  u^+(t) \\
  y^+(t)
\end{bmatrix} 
\equiv 
\begin{bmatrix}
  \Gamma \circ y^-(\tau) \\
  \Gamma \circ u^-(\tau)
\end{bmatrix},
$$

is such that $\bar{\Gamma}$ is an isometric isomorphism, satisfying the following

$$
\langle \xi, \zeta \rangle_{\mathcal{B}^-} = \langle \bar{\Gamma} \xi, \bar{\Gamma} \zeta \rangle_{\mathcal{B}^+}.
$$

Furthermore, $\bar{\Gamma}$ satisfies the following diagram:

$$
\begin{array}{c}
\mathcal{B}^- \\
\pi_{\mathcal{B}^-}
\end{array} \xrightarrow{\bar{\Gamma}} 
\begin{array}{c}
\mathcal{B}^+ \\
\pi_{\mathcal{B}^+}
\end{array}
\xrightarrow{\bar{\Gamma}^*} 
\begin{array}{c}
T\mathcal{B}^- \\
T\pi_{\mathcal{B}^-}
\end{array} \xrightarrow{\bar{\Gamma}^*} 
\begin{array}{c}
T\mathcal{B}^+ \\
T\pi_{\mathcal{B}^+}
\end{array}
$$

**Proof:** From definition of $\bar{\Gamma}$ in Eq. (19), by the isometric condition (15) it is verified that $\Gamma^\dagger \Gamma = I$ (in $\mathcal{T}\mathcal{B}^+$). Furthermore let $\hat{\alpha} \equiv \Gamma^* \zeta$ and $\beta \equiv \hat{\Gamma} \zeta$, then using Eq. (20) for $\Gamma^\dagger$, since the behavioral operator is a bijection,

$$
\langle \hat{\alpha}, \beta \rangle_{\mathcal{T}\mathcal{B}^-} = \langle \hat{\alpha}^\dagger, \beta^\dagger \rangle_{\mathcal{T}\mathcal{B}^+}.
$$

and this implies $\hat{\Gamma} \Gamma^* = I$ (in $\mathcal{T}\mathcal{B}^-$).

Moreover, since the tangent map of the behavioral operator is given by

$$
\tilde{\Gamma} \xi(w) \equiv \begin{bmatrix}
0 & \Gamma^\dagger \\
\Gamma & 0
\end{bmatrix} \begin{bmatrix}
\xi_u(w) \\
\xi_y(w)
\end{bmatrix},
$$

after substitution in Eq. (15), it implies that $\Gamma^\dagger \Gamma = I$ (in $\mathcal{U}$) and $\Gamma \Gamma^\dagger = I$ (in $\mathcal{Y}$).

The behavioral operator can be expressed in terms of the commutative diagram of Figure 1, such that

$$
\Gamma \circ u^-(\tau) \equiv \Psi_f \circ \Psi_p \circ u^-(\tau)
$$

$$
\Gamma^\dagger \circ y^-(\tau) \equiv \Psi_f^\dagger \circ \Psi_p^\dagger \circ y^-(\tau).
$$

where definitions for the operators $\Psi_f, \Psi_p, \Psi_f^\dagger$ and $\Psi_p^\dagger$ can be found in (Lopezlena, 2010, Def. 5.3, but their properties are irrelevant for the discussion of this paper.

**IV. NONLINEAR FACTORIZATION**

In this section we approach the topics of LQG balancing, coprime factorization and adjoint systems from our nonlinear behavioral-geometric viewpoint.

**A. LQG balancing**

The approach of LQG balancing by (Opdenacker and Jonckheere, 1985) admits an interpretation in the linear behavioral approach (Weiland, 1991). See (Scherpen and van der Schaft, 1994) for the nonlinear generalizations.

**Assumption 4.1:** The storage functions $S_r^*(x_0, r)$ (associated to system $\Sigma^-$) in Eq. (10) and $S_a(x_0, r)$ (associated to system $\Sigma^+$) in Eq. (11) exist and are well defined for the following specific supply rate $r(w(t))$, $r : \mathcal{W} \rightarrow \mathbb{R}^1$ defined by

$$
r(w(t)) \equiv \begin{bmatrix}
y(t) \\
u(t)
\end{bmatrix}^T \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
y(t) \\
u(t)
\end{bmatrix}.
$$

**Remark 4.1:** In the notation of (Scherpen and van der Schaft, 1994; Scherpen, 1994), the storage functions satisfying Assum. 4.1 are commonly called, the *backward-time past energy*, $K^-(x_0) \equiv S_a(x_0, r)$ and the future energy, $K^+(x_0) \equiv S_a(x_0, r)$.

**Example 4.1:** For the $\mathcal{L}$-class it is known (Weiland, 1991; Scherpen and van der Schaft, 1994), that associated to system (4) $K^-(x_0) = \frac{1}{2} x_0^T S x_0$ and associated to system (3) $K^+(x_0) = \frac{1}{2} x_0^T P x_0$, where the Gramians $S, P$ are stable solutions to the following algebraic Riccati Equations $AS + SA^T + BB^T = SC^TCS$ and $A^T P + PA + C^T C = BB^T P$ respectively.

The importance of supporting the nonlinear system $\Sigma^+$ in Eq. (1) by the Hilbert manifold framework of Assum. 2.1, 4.1 and 3.1 remains on the factibility of providing conditions for dissipative nonlinear balancing with the geometric machinery developed in (Lopezlena, 2008a; Lopezlena, 2008b).

Since Classical Curvature Theory characterizes the invariants of Riemannian submanifolds under isometric transformations, it is natural to use this tool to characterize the invariants of the behavior $\mathcal{B} \subset \mathcal{W} \times \mathcal{T}$, (Lopezlena, 2006a). Such invariants have equivalent invariants for $\mathcal{M} \times \mathcal{T}$, the following result compiles the relevant results in this regard.

**Corollary 4.1:** Let $\Sigma$ be supported by the Hilbert manifold structure, such that Eqs. (12)-(13) determine the past and future metrics by $\bar{g}_{\mathcal{M}^*(\hat{x}^*, 0)} \equiv \bar{g}_{\mathcal{B}^-}(\hat{w}^*, 0)$ and $\bar{g}_{\mathcal{M}^*(x^*, 0)} \equiv \bar{g}_{\mathcal{B}^+}(w^*, 0)$ respectively. The following is asserted:

1) There is a shape operator given by $A_{\eta}^{\mathcal{M}^*}(\alpha \cdot \cdot \cdot ) \equiv (\mathcal{T}^\dagger \alpha \cdot \cdot \cdot = (Q^T o P^T)\alpha \cdot \cdot \cdot = Q^T P^* \alpha \cdot \cdot \cdot$ and $\alpha \cdot \cdot \cdot \in T\mathcal{M}^\dagger, \eta \in (T\mathcal{M}^\dagger)^\dagger$, whose curvature can be
expressed by

\[
K(\alpha) = \frac{II_M(\alpha, \alpha)}{I_M(\alpha, \alpha)} = \frac{\langle T^*\alpha, \alpha \rangle_{T^*M_M}}{\langle \alpha, \alpha \rangle_{T^*M_M}}, \tag{22}
\]

s.t. \[I_M(\alpha, \alpha) = S_\alpha(x^0, r_T) \text{ and } II_M(\alpha, \alpha) = A^*_M(\alpha, \alpha)_{T^*M_M}. \]

2) A 1-form \( \beta \in T_M^* \), \( \langle \beta, \beta \rangle_{T^*M_M} = 1 \) is solution to the eigenvalue problem associated to \( K(\beta) \) in (22) if \( \beta \) is an eigenform of \( A^*_M \).

3) The set of eigenforms of \( A^*_M \), \( \{\beta_i | i = 1, \ldots, n; \beta_i \in T_M^* \} \), defines an orthonormal coframe of \( T_M^* \). Furthermore \( T_M^* \) can be locally spanned by a partition of eigencodistributions \( M^*_1 \oplus \cdots \oplus M^*_n \).

4) The frame \( \{\xi_i^+, \xi_i^- \} \) defined by dualization of the coframe in Cor. 4.1(3) spans \( T_M^* \).

5) Given a vectorfield \( \xi^+ \in T_M^* \), its local orthogonal projection \( \xi^+_{op} \in T_M^* \), defined in complementary subdomains \( M^*_1 \oplus \cdots \oplus M^*_n \). \( \xi^+_{op} \) is the nonlinear generalization to the adjoint system \( \Sigma^* \) of the backward-time system \( \Sigma^- \) in Eq. (2) mapped into forward-time. Therefore, under Assum. 4.1, the evolution-ary operator (semigroup) \( \hat{x}(t) = \Theta(T, 0, x^0, h^{-1} \circ \hat{y}(t)) \) describes the trajectories of \( \Sigma^*_\alpha \in (t, M^*, \Theta^*) \). Our Hilbert manifold framework is useful to support these systems.

Example 4.2: Let \( \Sigma^+ \) be the \( \mathcal{L} \)-class system (3) with forward-time response supported in \( (t, M, \Phi^T) \) and let \( \Sigma^- \) be the system (4) with backward-time response in \( (\tau, M^*, \Theta^T) \). The associated copo-adjoint system \( \Sigma^*_\alpha \equiv (t, \mathcal{M}^*, \Theta^T) \) is the \( \mathcal{L} \)-class adjoint system

\[
\Sigma^*_\alpha \colon \left\{ \begin{array}{l}
\dot{x}(t) = -AT \dot{x}(t) - CT \dot{y}(t), \\
\ddot{u}(t) = B^T \dot{x}(t),
\end{array} \right. \tag{27}
\]

Since in the \( \mathcal{L} \)-class the (Euclidean) state space \( \mathbb{R}^n \) can be identified at any point with its tangent space, i.e. \( \mathbb{R}^n \cong T_{x, t} \mathbb{R}^n \), Hilbert manifolds simplify to Euclidean Hilbert spaces and Lie group actions consist of linear maps. The unforced case is widely known, see e.g. (Skelton, 1988), in such case, \( \Sigma^+ \) and \( \Sigma^*_\alpha \) is such that \( \forall i \in t \nabla x_i \dot{x}_i k_0 = \sum_1^n x_i \dot{x}_i k, \quad x \in M, x \in M^*, k \in \mathbb{R}^1. \)

To conclude this section we provide a characterization of the nonlinear generalization to the adjoint system \( G(-s)^T \) as it is viewed in our geometric framework. Although nonlinear adjoint systems can be analyzed from the viewpoint of variational systems and hamiltonian extensions, see e.g. (Crouch and van der Schaft, 1987), our approach relies on the following abstract definition, extracted from (Lopezlena, 2008b):

**Definition 4.1 (Adjont system \( \Sigma^*_\alpha \)):** The \( \mathcal{N} \)-class dynamical system adjoint to \( \Sigma^+ \) is given by:

\[
\Sigma^*_\alpha \colon \left\{ \begin{array}{l}
\dot{x}(t) = \hat{f}(\hat{x}(t), h^{-1} \circ \hat{y}(t)), \\
\ddot{u}(t) = g^{-1}(\hat{x}(t)),
\end{array} \right. \tag{28}
\]

where \( \hat{x}(t) \in M^*, \hat{u}(t) \in U^* \) and \( \hat{y}(t) \in Y^* \). See definitions of the inverse maps \( h^{-1} : Y \mapsto M \), \( x(t) = h^{-1}(y(t)) \) and \( g^{-1} : M \mapsto U \), \( u(t) = g^{-1}(x(t)) \) detailed in (Lopezlena, 2008b).

**Definition 4.2:** Given two \( C_0 \)-semigroups \( \Phi^T(x) \) and \( \Theta^T(\hat{x}) \), defined in complement domain subdomains \( (\Phi(x, t), t \in \mathbb{R}^+ \) and \( (\Theta(\hat{x}, t), t \in \mathbb{R}^+ \)) on the state space \( \mathcal{M} \).
\(t, x \in D \subset M\) and \(\{\Theta(\hat{x}, \tau), \tau \in \tau, \hat{x} \in D^* \subset M^*\}\) s.t.

\[
\begin{align*}
\left[\Theta^t(\hat{x}^{-}(\tau))\right]^{-1} & = \Phi^t(x^+(t)), \\
\left[\nabla_x \Theta^t(\hat{x}^{-}(\tau))\right]^{-1}_{\tau=0} & = \nabla_x \Phi^t(x^+(t))|_{t=0},
\end{align*}
\]

is regular define their group extension defined by \(G(t, \tau)(x) \triangleq \Phi^t \circ \left[\Theta^t\right]^{-1} \circ x\), \(t \in \tau, \tau \in \tau, x \in D \subset M\).

**Example 4.3.** In the \(S\)-class, let the state transition matrix of the (unforced) system (3) be expressed as \(\Phi(t, \tau) = \psi(t)\psi^{-1}(\tau), \forall t, \tau \in (-\infty, \infty)\) for some nonsingular \(\psi(t)\) (a fundamental matrix). Furthermore, denote by \(\Theta(t, \tau)\) the adoint costate transition matrix associated to (the unforced) \(\Sigma^t_{+}\) in Eq. (27). Then it can be verified that \(\Theta^T(t, \tau) = \Phi(t, \tau) = \psi(t)\psi^{-1}(\tau), \forall t, \tau \in (-\infty, \infty)\) and thereby \(\Theta^T(t, \tau)\Phi(t, \tau) = I.\)

**Proposition 4.2.** Let the \(\Xi\)-class systems (1) and (2) satisfy Assum. 3.1 and 4.1. Let the group extensions associated to the \(\Xi\)-class systems (1) and (28) be denoted by

\[
\Phi(t, \tau)(x) = \Phi^t \circ \left[\Theta^t\right]^{-1} \circ x, \quad (30)
\]

\[
\Theta(t, \tau)(\hat{x}) = \Theta^t \circ \left[\Phi^t\right]^{-1} \circ \hat{x}, \quad (31)
\]

respectively. The following is asserted:

1) The semigroup actions are related by the diagram

\[
\begin{array}{c}
\xymatrix{
M_+ \ar@{-}[r]^-{\Phi} & M_0 \ar@{-}[r]^-{\Theta^t} & M_- \\
M'_+ \ar@{-}[r]^-{\Phi^t} & M'_0 \ar@{-}[r]^-{\Theta} & M'_-
}
\end{array}
\quad (32)
\]

2) The combination of operators (30) and (31) defines an isomomtry as in Def. 2.3.

**Proof:** 1) Diagram (32) follows directly from Fig. (1) considering only the internal spaces and semigroup actions instead of behavioral or exogenous variables.

2) Constructive. For every \(x_0 \in M\) there are two semitrajectories \(\sigma^+(x) \in M\) and \(\sigma^-(\hat{x}) \in M^*\) defined after the semigroup \(\sigma^+(x) = \Phi(t, 0, x_0, g \circ u^t)\) and \(\sigma^-(\hat{x}) = \Theta(t, 0, x_0, g \circ y^t)\) as expressed in diagram (32). Such semitrajectories are generated by \(\xi^+ \in TM\) and \(\beta^t \in T^*M\). Due to the Hilbert manifold structure, there is an isometric isomorphism \(E : M \simeq M^*\) such that \(||\beta^t||_{TM} = ||\xi^+||_{TM}\), implying \(\langle \beta^t, \beta^t \rangle_{TM} = \langle \xi^+, \xi^+ \rangle_{TM}\) and \(\langle \beta^t, \beta^t \rangle_{TM} = \langle \xi^+, \xi^* \rangle_{TM}\) is satisfied, see e.g. Table II in (Lopezlena, 2010). In consequence, for every initial condition \(x_0 \in M\) a future trajectory \(\sigma^+(x) \in M\) has a dual \(\sigma^+(x) \in M^*\) such that \(\sqrt{||\xi^+||_{TM}} = S_0(x_0)\) and a past trajectory \(\sigma^-(\hat{x}) \in M^*\) has a dual \(\sigma^- (x) \in M\) such that \(\sqrt{||\beta^t||_{TM}} = S_0(\hat{x}_0).\)

**REFERENCES**


