Global continuous control with desired gravity compensation for the finite-time and exponential stabilization of robot manipulators with constrained inputs

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Abstract: A global continuous control scheme for the finite-time or (local) exponential stabilization of robot manipulators with bounded inputs is developed involving *desired* gravity compensation. With respect to the *on-line* compensation case, the proposed controller entails a more complex closed-loop analysis, whence more involved requirements arise. Other important analytical limitations are further overcome through the developed algorithm. Computer simulations considering a robotic arm model corroborate the efficiency of the proposed controller.

Keywords: Finite-time stabilization; robot manipulators; desired gravity compensation; bounded inputs.

1. INTRODUCTION

A global continuous state-feedback scheme for the finite-time and exponential stabilization of mechanical/robotic systems with bounded inputs was developed and thoroughly motivated in (Zavala-Río and Zamora-Gómez, 2017). The explicit consideration of input constraints and the explicit choice on the system trajectory convergence are among the main characteristics that distinguish such an approach from continuous finite-time controllers developed for mechanical/robotic systems before its appearance: (Hong et al., 2002; Zhao et al., 2010; Sanyal and Bohn, 2015) (see for instance (Zavala-Río and Zamora-Gómez, 2017, §1) for a brief description of such works). Moreover, while the cited previous approaches mainly rely on the *dynamic inversion* technique (except for one of the two controllers presented in (Hong et al., 2002)), the scheme in (Zavala-Río and Zamora-Gómez, 2017) benefits from the inherent passive nature of mechanical systems. This is so in view of its (saturating) Proportional-Derivative type structure with exclusive compensation of the conservative-force (vector) term, which permits to reshape the closed-loop potential energy so as to set the desired posture as the only equilibrium position on the whole configuration space. The exclusive compensation of the conservative-force term allows to reduce the system model dependence of the designed scheme, consequently simplifying the control structure and decreasing the implied computation burden. However, such advantages could still be improved if the on-line compensation term could be replaced by the conservative-force/gravity term evaluated at the desired position. This idea was first introduced in (Takegaki and Arimoto, 1981) in an unconstrained-input conventional (infinite-time) stabilization framework and, since then, it has been well appreciated due to its simplicity and simplification improvements. This is the main motivation of this work, where a *desired-gravity-compensation* extension of the finite-time/exponential stabilization scheme from (Zavala-Río and Zamora-Gómez, 2017) is developed. Such a design task is not as simple or direct as a simple replacement of the *on-line* compensation term by the *desired* one, since the required (desired) closed-loop equilibrium position is kept, but not its uniqueness. Consequently, further design requirements prove to be needed to ensure that the control-induced potential energy component *dominates* the open-loop one (in order to solve the uniqueness issue). This was already pointed out in (Takegaki and Arimoto, 1981), where such a domination goal was shown to be achieved through a P control (vector) term with a stronger growing rate than that of the open-loop conservative force term in any direction (at every point) on the configuration space; in particular, under the simple consideration of uncoupled linear P and D control actions, this was shown to be achieved by simply fixing

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P gains higher than the highest (induced) norm value of the Jacobian matrix of the conservative force term (assuming that such a Jacobian matrix is bounded). But the solution of the referred uniqueness issue cannot be that simple in the analytical context considered here in view of the special functions involved to guarantee the achievement of the formulated stabilization goal. This represents an important analytical challenge to which this work succeeds to give a suitable solution for robot manipulators with bounded inputs. The desired-gravity-compensation developed approach implies an important simplification on the control implementation with respect to the on-line compensation version. Its efficiency is further corroborated through simulation results.

2. PRELIMINARIES

Let $X \in \mathbb{R}^{m\times n}$ and $y \in \mathbb{R}^n$. X_{ij} stands for the element of X at its i^{th} row and j^{th} column, X_i for the i^{th} row of X and y_i for the i^{th} element of y. With $m = n$, $X > 0$ denotes that X is positive definite; for a symmetric matrix X, $\lambda_m(X)$ and $\lambda_M(X)$ respectively stand for its minimum and maximum eigenvalues. As conventionally, for sets A and B, $A \setminus B$ represents the subset of elements that are in A and are not in B. 0_n represents the origin of $\mathbb{R}^n$ and I_n the $n \times n$ identity matrix. $\mathbb{R}^n_{>0}$ and $\mathbb{R}^n_{\geq 0}$ denote the set of n-tuples with positive and non-negative entries, respectively. $\|\cdot\|$ stands for the standard Euclidean norm for vectors and induced norm for matrices. Let $S_c^{n-1} = \{x \in \mathbb{R}^n : \|x\| = c\}$: an $(n-1)$-dimensional sphere of radius $c > 0$ on $\mathbb{R}^n$. We denote $D_g f$ the directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ along $g : \mathbb{R}^n \to \mathbb{R}^n$, *i.e.*, $D_g f(x) = \frac{\partial f}{\partial x} g(x)$. We consider the sign function $\text{sign}(\cdot)$ to be zero at zero, and $\text{sat}(\varsigma) = \text{sign}(\varsigma)\min\{|\varsigma|, 1\}$.

2.1 Robot manipulators

Consider the n-degree-of-freedom (DOF) fully-actuated robot manipulator dynamics

$$H(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau \tag{1}$$

where q, $\dot{q}$, $\ddot{q} \in \mathbb{R}^n$ are the position (generalized coordinates), velocity, and acceleration vectors; the inertial matrix $H(q) \in \mathbb{R}^{n\times n}$ is a continuously differentiable positive definite symmetric matrix function, such that $H(q) \geq \mu_m I_n$, $\forall q \in \mathbb{R}^n$, for some $\mu_m > 0$; the Coriolis and centrifugal effect matrix $C(q, \dot{q}) \in \mathbb{R}^{n\times n}$, defined through the Christoffel symbols of the first kind, satisfies $\dot{H}(q, \dot{q}) = C(q, \dot{q}) + C^T(q, \dot{q})$, $\forall q, \dot{q} \in \mathbb{R}^n$, and consequently

$$z^T \left[\frac{1}{2}\dot{H}(x, y) - C(x, y)\right] z = 0 \tag{2a}$$

$\forall x, y, z \in \mathbb{R}^n$, where $\dot{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, with $\dot{H}_{ij}(q, \dot{q}) = \frac{\partial H_{ij}}{\partial q}(q)\dot{q}$, $i, j = 1, \ldots, n$,

$$\|C(x, y)\| \leq \psi(x)\|y\| \tag{2b}$$

for some $\psi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and $C(x, y)z = C(x, z)y$, $\forall x, y, z \in \mathbb{R}^n$, whence we have that

$$C(q, a\dot{q})b\dot{q} = C(q, b\dot{q})a\dot{q} = C(q, ab\dot{q})\dot{q} = C(q, \dot{q})ab\dot{q} \tag{3}$$

$\forall q, \dot{q} \in \mathbb{R}^n$, $\forall a, b \in \mathbb{R}$; $g(q) = \nabla \mathcal{U}_{\text{ol}}(q)$, with $\mathcal{U}_{\text{ol}} : \mathbb{R}^n \to \mathbb{R}$ being the potential energy due to gravity, or equivalently $\mathcal{U}_{\text{ol}}(q) = \mathcal{U}_{\text{ol}}(q_0) + \int_{q_0}^{q} g^T(z)dz$, for any $q, q_0 \in \mathbb{R}^n$; and $\tau \in \mathbb{R}^n$ is the external input (generalized) force vector. We consider the (realistic) bounded input case, where each input τ_i is constrained by a saturation bound $T_i > 0$. More precisely, letting u_i represent the control variable (controller output) relative to the i^{th} degree of freedom, we have that

$$\tau_i = T_i \text{sat}(u_i/T_i) \tag{4}$$

Assumption 2.1. $H(q)$ is bounded, *i.e.* $\|H(q)\| \leq \mu_M$, $\forall q \in \mathbb{R}^n$, for some $\mu_M \geq \mu_m > 0$.

Assumption 2.2. $\psi(\cdot)$ in (2b) is bounded and consequently $\|C(x, y)\| \leq k_C\|y\|$, $\forall x, y \in \mathbb{R}^n$, for some $k_C \geq 0$.

Assumption 2.3. The gravity force vector is a continuously differentiable bounded vector function with bounded Jacobian matrix $\frac{\partial g}{\partial q}$, or equivalently: $|g_i(q)| \leq B_{gi}$, $\forall q \in \mathbb{R}^n$, for some non-negative constant B_{gi}; $\left\|\frac{\partial g}{\partial q}(q)\right\| \leq k_g$, $\forall q \in \mathbb{R}^n$, for some non-negative constant k_g, and consequently $\|g(x) - g(y)\| \leq k_g\|x - y\|$, $\forall x, y \in \mathbb{R}^n$.

Assumption 2.4. $T_i > \eta B_{gi}$, $i = 1, \ldots, n$, with $\eta \geq 1$.

Assumptions 2.1–2.3 apply *e.g.* for robots having only revolute joints (Kelly et al., 2005, §4.3).

2.2 Local homogeneity, finite-time/δ-exponential stability

This work is developed within the analytical framework of *local homogeneity* (Zavala-Río and Fantoni, 2014), which states a formal analytical platform permitting to handle vector fields with bounded components. Definitions and results in such an analytical context are strongly related to *family of dilations* δ_ε^r, defined as $\delta_\varepsilon^r(x) = \left(\varepsilon^{r_1}x_1, \ldots, \varepsilon^{r_n}x_n\right)^T$, $\forall x \in \mathbb{R}^n$, $\forall \varepsilon > 0$, with $r = (r_1, \ldots, r_n)^T$, where the *dilation coefficients* $r_1, \ldots, r_n$ are positive scalars. Other fundamental concepts involved in the analytical context underlying this work are those of *homogeneous norm* —with respect to the family of dilations δ_ε^r, or simply *r-homogeneous norm*: a positive definite continuous function being r-homogeneous of degree 1— (Zavala-Río and Zamora-Gómez, 2017), denoted $\|\cdot\|_r$, and *r-homogeneous* $(n-1)$*-sphere* of radius $c > 0$: $S_{r,c}^{n-1} = \{x \in \mathbb{R}^n : \|x\|_r = c\}$.

Consider an n-th order autonomous system

$$\dot{x} = f(x) \tag{5}$$

where f is a vector field being continuous on an open neighborhood of the origin $\mathcal{D} \subset \mathbb{R}^n$ and such that $f(0_n) = 0_n$, and let $x(t; x_0)$ represent the system solution with initial condition $x(0; x_0) = x_0$. An important definition for this work is that of a *finite-time stable* equilibrium as stated in (Bhat and Bernstein, 2005).

Remark 2.1. The origin is a globally finite-time stable equilibrium of system (5) if and only if it is globally asymptotically stable and finite-time stable. △

Theorem 2.1. (Zavala-Río and Fantoni, 2014) Consider system (5) with $\mathcal{D} = \mathbb{R}^n$. Suppose that f is a locally r-homogeneous vector field of degree α with domain of homogeneity $D \subset \mathbb{R}^n$. Then, the origin is a globally finite-time stable equilibrium of system (5) if and only if it is globally asymptotically stable and $\alpha < 0$.

An alternative stability concept proving to be compatible to the framework of (local) homogeneity is that of δ-*exponential stability*, whose definition is found for instance in (Zavala-Río and Zamora-Gómez, 2017).

Remark 2.2. If f in (5) is locally r-homogeneous of degree $\alpha = 0$ with dilation coefficients $r_i = r_0$, $\forall i \in \{1, \dots, n\}$, for some $r_0 > 0$, then the origin turns out to be exponentially stable (in the standard sense (Khalil, 2002, Definition 4.5)) if and only if it is δ-exponentially stable (Zavala-Río and Zamora-Gómez, 2017, Remark 2.5). △

Consider an n-th order autonomous system of the form

$$\dot{x} = f(x) + \hat{f}(x) \tag{6}$$

where f and $\hat{f}$ are continuous vector fields on $\mathbb{R}^n$ such that $f(0_n) = \hat{f}(0_n) = 0_n$.

Lemma 2.1. (Zavala-Río and Zamora-Gómez, 2017, Lemma 2.2) Suppose that, for some $r \in \mathbb{R}^n_{>0}$, f in (6) is a locally r-homogeneous vector field of degree $\alpha < 0$, resp. $\alpha = 0$, with domain of homogeneity $D \subset \mathbb{R}^n$, and that 0_n is a globally asymptotically, resp. δ-exponentially, stable equilibrium of $\dot{x} = f(x)$. Then, the origin is a finite-time, resp. δ-exponentially, stable equilibrium of system (6) if

$$\lim_{\varepsilon \to 0^+} \frac{\hat{f}_i(\delta^r_\varepsilon(x))}{\varepsilon^{\alpha + r_i}} = 0$$

$i = 1, \dots, n$, $\forall x \in S^{n-1}_c$, resp. $\forall x \in S^{n-1}_{r,c}$, for some $c > 0$ such that $S^{n-1}_c \subset D$, resp. $S^{n-1}_{r,c} \subset D$.

Remark 2.3. The condition required by Lemma 2.1 may be equivalently verified through the satisfaction of

$$\lim_{\varepsilon \to 0^+} \left\| \varepsilon^{-\alpha} \mathrm{diag}\left[\varepsilon^{-r_1}, \dots, \varepsilon^{-r_n}\right] \hat{f}(\delta^r_\varepsilon(x)) \right\| = 0$$

$\forall x \in S^{n-1}_c$ (resp. $S^{n-1}_{r,c}$). △

2.3 Scalar functions with particular properties

Definition 2.1. A continuous scalar function $\sigma : \mathbb{R} \to \mathbb{R}$ will be said to be:

(1) bounded —by M— if $|\sigma(\varsigma)| \leq M$, $\forall \varsigma \in \mathbb{R}$, for some positive constant M;
(2) strictly passive if $\varsigma\sigma(\varsigma) > 0$, $\forall \varsigma \neq 0$;
(3) strongly passive if it is a strictly passive function satisfying $|\sigma(\varsigma)| \geq \kappa \left| a \,\mathrm{sat}(\varsigma/a) \right|^b = \kappa \big(\min\{|\varsigma|, a\}\big)^b$, $\forall \varsigma \in \mathbb{R}$, for some positive constants κ, a and b.

Remark 2.4. Equivalent characterizations of strictly passive functions are: $\varsigma\sigma(\varsigma) > 0 \iff \mathrm{sign}(\varsigma)\sigma(\varsigma) > 0 \iff \mathrm{sign}(\sigma(\varsigma)) = \mathrm{sign}(\varsigma)$, $\forall \varsigma$. △

Lemma 2.2. (Zavala-Río and Zamora-Gómez, 2017, Lemma 2.3) Let $\sigma : \mathbb{R} \to \mathbb{R}$, $\sigma_0 : \mathbb{R} \to \mathbb{R}$ and $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be strongly passive functions and k be a positive constant. Then:

(1) $\int_0^\varsigma \sigma(k\nu) d\nu > 0$, $\forall \varsigma \neq 0$;
(2) $\int_0^\varsigma \sigma(k\nu) d\nu \to \infty$ as $|\varsigma| \to \infty$;
(3) $\sigma_0 \circ \sigma_1$ is strongly passive.

3. THE PROPOSED CONTROL SCHEME

Consider the Saturating-Proportional Saturating-Derivative type controller with *desired* gravity compensation given as

$$u(q, \dot{q}) = -s_1(K_1 \bar{q}) - s_2(K_2 \dot{q}) + g(q_d) \tag{7}$$

where $\bar{q} = q - q_d$, for any constant (desired equilibrium position) $q_d \in \mathbb{R}^n$; $K_i = \mathrm{diag}[k_{i1}, \dots, k_{in}]$, $i = 1, 2$, are positive definite diagonal matrices —*i.e.* $K_i = \mathrm{diag}[k_{i1}, \dots, k_{in}]$, $k_{ij} > 0$, $i = 1, 2$, $j = 1, \dots, n$— with K_1 involved in an additional requirement stated below (through (9)); for any $x \in \mathbb{R}^n$, $s_i(x) = \big(\sigma_{i1}(x_1), \dots, \sigma_{in}(x_n)\big)^T$, $i = 1, 2$, with, for each $j = 1, \dots, n$, σ_{ij} being non-decreasing strictly passive functions, such that

$$B_j \triangleq \max\left\{ \lim_{\varsigma \to \infty} \sigma_{0j}(\varsigma) \, , \, \lim_{\varsigma \to -\infty} -\sigma_{0j}(\varsigma) \right\} < T_j - B_{gj} \tag{8}$$

with $\sigma_{0j}(\varsigma) = \sigma_{1j}(\varsigma) + \sigma_{2j}(\varsigma)$, both being locally Lipschitz-continuous on $\mathbb{R} \setminus \{0\}$; and with, for each $j = 1, \dots, n$, k_{1j} and σ_{1j} additionally required to be such that, for all $\varsigma \in \mathbb{R}$,

$$\left|\sigma_{1j}(k_{1j}\varsigma)\right| > \min\left\{k_g|\varsigma| \, , \, 2B_{gj}\right\} \tag{9}$$

Remark 3.1. From the above formulation, we have that

$$2B_{gj} < |\sigma_{1j}(k_{1j}\varsigma)| \leq B_j < T_j - B_{gj}$$

$\forall |\varsigma| \geq 2B_{gj}/k_g$, whence one sees that Assumption 2.4 with $\eta = 3$ is a necessary condition for the feasibility of the simultaneous fulfilment of (8) and (9). △

Remark 3.2. Inequality (9) implies the existence of constants $\hat{k}_{1j} > k_g$ and $b_j > 2B_{gj}$ such that $\left|\sigma_{1j}(k_{1j}\varsigma)\right| \geq \min\left\{\hat{k}_{1j}|\varsigma|, b_j\right\} > \min\left\{k_g|\varsigma|, 2B_{gj}\right\}$, $\forall \varsigma \neq 0$. △

Proposition 3.1. Consider system (1),(4) in closed loop with the proposed control law (7), under Assumptions 2.1–2.3 and 2.4 with $\eta = 3$, and the above stated design specifications. Thus, global asymptotic stability of the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is guaranteed with $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \dots, n$, $\forall t \geq 0$.

Proof. Observe that —for every $j = 1, \dots, n$— by (8), we have that, for any $(q, \dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$ and any $q_d \in \mathbb{R}^n$: $|u_j(q, \dot{q})| \leq \left|\sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(k_{2j}\dot{q}_j)\right| + \left|g_j(q_d)\right| \leq B_j + B_{gj} < T_j$. From this and (4), one sees that $T_j > |u_j(q, \dot{q})| = |u_j| = |\tau_j|$, $\forall (q, \dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$, which shows that, along the system trajectories, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \dots, n$, $\forall t \geq 0$. Hence, the closed-loop dynamics takes the form

$$H(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = -s_1(K_1\bar{q}) - s_2(K_2\dot{q}) + g(q_d)$$

By defining $x_1 = \bar{q}$, $x_2 = \dot{q}$, and $x = (x_1^T, x_2^T)^T$ the closed-loop dynamics adopts the form of (6) with

$$f(x) = \begin{pmatrix} f_{(1)}(x) \\ f_{(2)}(x) \end{pmatrix} \quad , \quad \hat{f}(x) = \begin{pmatrix} \hat{f}_{(1)}(x) \\ \hat{f}_{(2)}(x) \end{pmatrix} \tag{10}$$

where $f_{(1)}(x) = x_2$, $f_{(2)}(x) = -H^{-1}(q_d)\big[s_1(K_1x_1) + s_2(K_2x_2)\big]$, $\hat{f}_{(1)}(x) = 0_n$, and

$$\begin{aligned} \hat{f}_{(2)}(x) = -H(x_1 + q_d)[C(x_1 + q_d, x_2)x_2 + g(x_1 + q_d) - g(q_d)] \\ - \mathcal{H}(x_1)[s_1(K_1x_1) + s_2(K_2x_2)] \end{aligned} \tag{11}$$

with

$$\mathcal{H}(x_1) = H^{-1}(x_1 + q_d) - H^{-1}(q_d) \tag{12}$$

Thus, the closed-loop stability property stated through Proposition 3.1 is corroborated by showing that $x = 0_{2n}$ is a globally asymptotically stable equilibrium of the state equation $\dot{x} = f(x) + \hat{f}(x)$, which is proven through the following theorem. □

Theorem 3.1. Under the stated specifications, the origin is a globally asymptotically stable equilibrium of $\dot{x} = f(x) + \ell \hat{f}(x)$, $\forall \ell \in \{0, 1\}$, with $f(x)$ and $\hat{f}(x)$ defined through Eqs. (10).

Proof. For every $\ell \in \{0, 1\}$, let us define the continuously differentiable scalar function

$$V_\ell(x_1,x_2)=\frac{1}{2}x_2^T H(\ell x_1+q_d)x_2+\mathcal{U}_\ell(x_1) \tag{13}$$

where

$$\mathcal{U}_\ell(x_1)\triangleq\int_{0_n}^{x_1}s_1^T(K_1z)dz+\ell\,\mathcal{U}(x_1) \tag{14}$$

$\int_{0_n}^{x_1}s_1^T(K_1z)dz=\sum_{j=1}^n\int_0^{x_{1j}}\sigma_{1j}(k_{1j}z_j)dz_j$,

$$\mathcal{U}(x_1)\triangleq\mathcal{U}_{\rm ol}(x_1+q_d)-\mathcal{U}_{\rm ol}(q_d)-g^T(q_d)x_1 \tag{15a}$$

$$=\int_{0_n}^{x_1}\big[g(z+q_d)-g(q_d)\big]^T dz \tag{15b}$$

$$=\int_{0_n}^{x_1}\left[\int_{0_n}^{z}\frac{\partial g}{\partial q}(\bar z+q_d)d\bar z\right]^T dz \tag{15c}$$

Observe from Eqs. (15) and Assumption 2.3 that

$$\mathcal{U}(x_1)\le\int_{0_n}^{x_1}\left[\int_{0_n}^{z}\left\|\frac{\partial g}{\partial q}(\bar z+q_d)\right\|d\bar z\right]^T dz$$

$$\le\int_{0_n}^{x_1}k_g z^T dz=\sum_{j=1}^n\int_0^{x_{1j}}k_g z_j dz_j \tag{16}$$

$\forall x_1\in\mathbb{R}^n$ (from (15c)), and simultaneously that

$$\mathcal{U}(x_1)\le\sum_{j=1}^n\int_0^{x_{1j}}\mathrm{sign}(z_j)\big|g_j(z+q_d)-g_j(q_d)\big|dz_j$$

$$\le\sum_{j=1}^n\int_0^{x_{1j}}\mathrm{sign}(z_j)2B_{gj}dz_j$$

$\forall x_1\in\mathbb{R}^n$ (from (15b)). From these inequalities, the satisfaction of (9), and Remark 3.2, we have that

$$\mathcal{U}_\ell(x_1)\ge\sum_{j=1}^n\int_0^{x_{1j}}\mathrm{sign}(z_j)\min\Big\{(\hat k_{1j}-\ell k_g)|z_j|\,,\ (b_j-2\ell B_{gj})\Big\}dz_j$$

$$=\sum_{j=1}^n w_{\ell j}(x_{1j})\triangleq S_\ell(x_1) \tag{17a}$$

with

$$w_{\ell j}(x_{1j})=\begin{cases}\dfrac{\bar k_{\ell j}}{2}x_{1j}^2 & \text{if } |x_{1j}|\le\bar b_{\ell j}/\bar k_{\ell j}\\ \bar b_{\ell j}\big[|x_{1j}|-\bar b_{\ell j}/(2\bar k_{\ell j})\big] & \text{if } |x_{1j}|>\bar b_{\ell j}/\bar k_{\ell j}\end{cases} \tag{17b}$$

for some $\hat k_{1j}>k_g$ and $b_j>2B_{gj}$, and any positive constants $\bar k_{\ell j}\le\hat k_{1j}-\ell k_g$ and $\bar b_{\ell j}\le b_j-2\ell B_{gj}$.

Remark 3.3. Note from expressions (17) that $S_\ell(x_1)$, $\ell=0,1$, are positive definite radially unbounded functions. Observe further that (involving previous arguments and Remark 2.4)

$$D_{x_1}\mathcal{U}_\ell(x_1)=x_1^T\Big[s_1(K_1x_1)+\ell\big(g(x_1+q_d)-g(q_d)\big)\Big]$$

$$\ge\sum_{j=1}^n|x_{1j}|\Big[\big|\sigma_{1j}(k_{1j}x_{1j})\big|-\ell\big|g_j(x_1+q_d)-g_j(q_d)\big|\Big]$$

$$\ge\sum_{j=1}^n|x_{1j}|\min\big\{\bar k_{\ell j}|x_{1j}|\,,\ \bar b_{\ell j}\big\}>0 \tag{18}$$

$\forall x_1\ne 0_n$, whence one sees that, for every $\ell=0,1$, $\nabla_{x_1}\mathcal{U}_\ell(x_1)=s_1(K_1x_1)+\ell\big[g(x_1+q_d)-g(q_d)\big]=0_n\iff x_1=0_n$ △

Thus, from Eqs. (13) and (17) and the properties of $H(q)$ we get that

$$V_\ell(x_1,x_2)\ge\frac{\mu_m}{2}\|x_2\|^2+S_\ell(x_1) \tag{19}$$

whence positive definiteness and radial unboundedness of V_ℓ, $\ell=0,1$, is concluded. Further, for every $\ell\in\{0,1\}$, the derivative of V_ℓ along the trajectories of $\dot x=f(x)+\ell\hat f(x)$, is obtained as $\dot V_\ell(x_1,x_2)=x_2^TH(\ell x_1+q_d)\dot x_2+\frac{\ell}{2}x_2^T\dot H(x_1+q_d,x_2)x_2+\big[s_1(K_1x_1)+\ell[g(x_1+q_d)-g(q_d)]\big]^T\dot x_1=x_2^T\big[-\ell[C(x_1+q_d,x_2)x_2+g(x_1+q_d)-g(q_d)]-s_1(K_1x_1)-s_2(K_2x_2)\big]+\frac{\ell}{2}x_2^T\dot H(x_1+q_d,x_2)x_2+\big[s_1(K_1x_1)+\ell[g(x_1+q_d)-g(q_d)]\big]^Tx_2=-x_2^Ts_2(K_2x_2)=-\sum_{j=1}^n x_{2j}\sigma_{2j}(k_{2j}x_{2j})$ where, in the case of $\ell=1$, (2a) has been applied. Notice, from the strictly passive character of σ_{2j}, that $\dot V_\ell(x_1,x_2)\le 0$, $\forall(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n$, with $Z_\ell\triangleq\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n:\dot V_\ell(x_1,x_2)=0\}=\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n:x_2=0_n\}$. Further, from the system dynamics $\dot x=f(x)+\ell\hat f(x)$ —due to the positive definiteness of H and Remark 3.3— one sees that $x_2(t)\equiv 0_n\implies\dot x_2(t)\equiv 0_n\implies s_1(K_1x_1(t))+\ell\big[g\big(x_1(t)+q_d\big)-g(q_d)\big]\equiv 0_n\iff x_1(t)\equiv 0_n$ (which shows that $(x_1,x_2)(t)\equiv(0_n,0_n)$ is the only system solution completely remaining in Z_ℓ), and corroborates that at any $(x_1,x_2)\in\{(\bar q,\dot q)\in Z_\ell:\bar q\ne 0_n\}$, the resulting unbalanced force terms act on the closed-loop dynamics $\big[\dot x=f(x_1,0_n)+\ell\hat f(x_1,0_n)$ with $(x_1,x_2)\ne(0_n,0_n)\big]$, forcing the system trajectories to leave Z_ℓ, whence $\{(0_n,0_n)\}$ is concluded to be the only invariant set in Z_ℓ, $\ell=0,1$. Therefore, by the invariance theory (Michel et al., 2008, §7.2), $x=0_{2n}$ is concluded to be a globally asymptotically stable equilibrium of both the state equation $\dot x=f(x)$ and the (closed-loop) system $\dot x=f(x)+\hat f(x)$. □

3.1 Finite-time/exponential stabilization

Proposition 3.2. Consider the proposed control scheme under the additional consideration that, for every $j=1,\dots,n$, σ_{ij}, $i=1,2$, are locally r_i-homogeneous of degree $\alpha_j=2r_2-r_1$ —i.e. $r_{1j}=r_1$, $r_{2j}=r_2$ and $\alpha_{1j}=\alpha_{2j}=\alpha_j=2r_2-r_1$ for all $j=1,\dots,n$— with domain of homogeneity $D_{ij}=\{\varsigma\in\mathbb{R}:|\varsigma|<L_{ij}\in(0,\infty]\}$, for some dilation coefficients $r_i>0$, $i=1,2$, such that $\alpha_j=2r_2-r_1>0>r_2-r_1$. Thus, global finite-time stability of the closed-loop trivial solution $\bar q(t)\equiv 0_n$ is guaranteed with $|\tau_j(t)|=|u_j(t)|<T_j$, $j=1,\dots,n$, $\forall t\ge 0$.

Proof. Note that Proposition 3.1 holds and consequently $|\tau_j(t)|=|u_j(t)|<T_j$, $j=1,\dots,n$, $\forall t\ge 0$. Then, all that remains to be proven is that the additional considerations give rise to finite-time stabilization. In this direction, let $\hat r_i=(r_{i1},\dots,r_{in})^T$, $i=1,2$, $r=(\hat r_1^T,\hat r_2^T)^T$, $D\triangleq\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n:K_ix_i\in D_{i1}\times\cdots\times D_{in}\,,\ i=1,2\}=\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n:|x_{1j}|<L_{1j}/k_{1j},|x_{2j}|<L_{2j}/k_{2j}\ ,j=1,\dots,n\}$, and consider the previously defined state (vector) variables and the consequent closed-loop state-space representation $\dot x=f(x)+\hat f(x)$, with f and $\hat f$ as defined through Eqs. (10). Since D defines an open neighborhood of the origin, there exists $\rho>0$ such that $B_\rho\triangleq\{x\in\mathbb{R}^{2n}:\|x\|<\rho\}\subset D$. Moreover, for every $x\in B_\rho$ and all $\varepsilon\in(0,1]$, we have that $\delta_\varepsilon^r(x)\in B_\rho$ (since $\|\delta_\varepsilon^r(x)\|<\|x\|$, $\forall\varepsilon\in(0,1)$), and, for every $j\in\{1,\dots,n\}$,

$$f_j(\delta_\varepsilon^r(x)) = \varepsilon^{r_{2j}} x_{2j} = \varepsilon^{(r_2-r_1)+r_{1j}} f_j(x)$$

$$\begin{aligned} f_{n+j}(\delta_\varepsilon^r(x)) &= -H_j^{-1}(q_d)\big(s_1(\varepsilon^{r_1}K_1x_1) + s_2(\varepsilon^{r_2}K_2x_2)\big) \\ &= -\varepsilon^{\alpha_1} H_j^{-1}(q_d)\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \\ &= \varepsilon^{(r_2-r_1)+r_{2j}} f_{n+j}(x) \end{aligned}$$

whence one sees that f is a locally r-homogeneous vector field of degree $\alpha = r_2 - r_1$, with domain of homogeneity B_ρ. Hence, by Theorems 2.1 and 3.1, the origin of $\dot{x} = f(x)$ is concluded to be a globally finite-time stable equilibrium since $r_2 - r_1 < 0$. Thus, by Theorem 3.1, Lemma 2.1, and Remarks 2.1 and 2.3, we conclude that the origin of the closed-loop system $\dot{x} = f(x) + \hat{f}(x)$ is a global finite-time stable equilibrium, provided that $r_2 - r_1 < 0$, if

$$\begin{aligned} \mathcal{L}_0 &\triangleq \lim_{\varepsilon\to 0^+} \left\| \varepsilon^{-\alpha} \delta_\varepsilon^{-\hat{r}_2}\big(\hat{f}_{(2)}(\delta_\varepsilon^r(x))\big)\right\| \\ &= \lim_{\varepsilon\to 0^+} \left\| \varepsilon^{-\alpha-r_2} \hat{f}_{(2)}(\delta_\varepsilon^r(x))\right\| \\ &= \lim_{\varepsilon\to 0^+} \varepsilon^{r_1-2r_2} \left\| \hat{f}_{(2)}(\delta_\varepsilon^r(x))\right\| = 0 \end{aligned} \tag{20}$$

for all $x \in S_c^{2n-1} = \{x \in \mathbb{R}^{2n} : \|x\| = c\}$, for some $c > 0$ such that $S_c^{2n-1} \subset D$. Hence, under the consideration of (11) and (3), we have for all such $x \in S_c^{2n-1}$:

$$\begin{aligned} &\left\| \hat{f}_{(2)}(\delta_\varepsilon^r(x))\right\| \\ &\quad \le \left\| H^{-1}(\varepsilon^{r_1}x_1 + q_d) C(\varepsilon^{r_1}x_1 + q_d, x_2)\varepsilon^{2r_2}x_2\right\| \\ &\qquad + \left\| H^{-1}(\varepsilon^{r_1}x_1 + q_d)\right\| \left\| g(\varepsilon^{r_1}x_1 + q_d) - g(q_d)\right\| \\ &\qquad + \left\| \mathcal{H}(\varepsilon^{r_1}x_1)\big[\varepsilon^{\alpha_1}s_1(K_1x_1) + \varepsilon^{\alpha_2}s_2(K_2x_2)\big]\right\| \\ &\quad \le \varepsilon^{2r_2}\left\| H^{-1}(\varepsilon^{r_1}x_1 + q_d) C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2\right\| \\ &\qquad + \left\| H^{-1}(\varepsilon^{r_1}x_1 + q_d)\right\| k_g \varepsilon^{r_1}\|x_1\| \\ &\qquad + \varepsilon^{2r_2-r_1}\left\| \mathcal{H}(\varepsilon^{r_1}x_1)\right\| \cdot \left\| s_1(K_1x_1) + s_2(K_2x_2)\right\| \end{aligned}$$

and consequently, from (20) (recall that $r_1 > r_2 > 0$),

$$\begin{aligned} \mathcal{L}_0 &\le \lim_{\varepsilon\to 0^+} \varepsilon^{r_1}\left\| H^{-1}(\varepsilon^{r_1}x_1 + q_d) C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2\right\| \\ &\quad + k_g\|x_1\| \lim_{\varepsilon\to 0^+} \varepsilon^{2(r_1-r_2)}\left\| H^{-1}(\varepsilon^{r_1}x_1 + q_d)\right\| \\ &\quad + \lim_{\varepsilon\to 0^+} \left\| \mathcal{H}(\varepsilon^{r_1}x_1)\right\| \cdot \left\| s_1(K_1x_1) + s_2(K_2x_2)\right\| \\ &\le \left\| H^{-1}(q_d)C(q_d, x_2)x_2\right\| \lim_{\varepsilon\to 0^+} \varepsilon^{r_1} \\ &\quad + k_g\|x_1\|\left\| H^{-1}(q_d)\right\| \lim_{\varepsilon\to 0^+} \varepsilon^{2(r_1-r_2)} \\ &\quad + \left\| s_1(K_1x_1) + s_2(K_2x_2)\right\| \lim_{\varepsilon\to 0^+} \left\| \mathcal{H}(\varepsilon^{r_1}x_1)\right\| \\ &\le \left\| s_1(K_1x_1) + s_2(K_2x_2)\right\| \cdot \left\| \mathcal{H}(0_n)\right\| = 0 \end{aligned} \tag{21}$$

(note, from (12), that $\|\mathcal{H}(0_n)\| = \|H^{-1}(q_d) - H^{-1}(q_d)\| = 0$), which completes the proof. □

Corollary 3.1. Consider the proposed control scheme taking σ_{ij}, $i = 1, 2$, $j = 1, \ldots, n$, such that

$$\sigma_{ij}(\varsigma) = \operatorname{sign}(\varsigma)|\varsigma|^{\beta_{ij}} \quad \forall |\varsigma| \le L_{ij} \in (0, \infty) \tag{22}$$

with constants β_{ij} such that

$$0 < \beta_{1j} \le 1 \quad , \quad \beta_{2j} = \frac{2\beta_{1j}}{1 + \beta_{1j}} \tag{23}$$

Thus, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \ge 0$, and the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is:

(1) globally finite-time stable if $0 < \beta_1 < 1$;
(2) globally asymptotically stable and (locally) exponentially stable if $\beta_1 = 1$.

Item 1 of Corollary 3.1 is proven by corroborating that, under the stated conditions, for every $j = 1, \ldots, n$ and any $r_1 > 0$, by taking $r_{1j} = r_1$ and $r_{2j} = r_2 = (1 + \beta_1)r_1/2$, the requirements of Proposition 3.2 are satisfied with $0 < \beta_1 < 1 \implies r_2 - r_1 < 0 < 2r_2 - r_1$. On the other hand, note that with $r_2 = r_1$ —or analogously $\beta_1 = 1$ in the context of Corollary 3.1— we have that $\varepsilon^{r_2 - r_1} = 1$, $\forall \varepsilon > 0$. Hence, in this case, developments analog to those giving rise to inequalities (21) lead to $\mathcal{L}_0 \le k_g\|x_1\|\|H^{-1}(q_d)\|$, and consequently, Lemma 2.1 (under the consideration of Remark 2.2) cannot be applied to conclude (local) exponential stability (contrarily to the on-line conservative-force compensation case of (Zavala-Río and Zamora-Gómez, 2017)). However, while the global asymptotic stability follows from Proposition 3.1, the (local) exponential stability stated through item 2 of Corollary 3.1 is proven by showing that, for a sufficiently small value of ε,

$$V_2(x_1, x_2) = V_1(x_1, x_2) + \epsilon x_1^T H(x_1 + q_d)x_2$$

—with V_1 as defined through Eq. (13)— is a suitable strict Luapunov function of the closed-loop system, on a neighborhood of the origin 0_{2n}. In particular, with

$$\epsilon < \min\{\epsilon_1, \epsilon_2\}$$

$$\epsilon_1 = \frac{\big[\bar{k}_{1m}\mu_m\big]^{1/2}}{\mu_M} \quad , \quad \epsilon_2 = \frac{\bar{k}_{1m}k_{2m}}{\bar{k}_{1m}k_C\varrho + \bar{k}_{1m}\mu_M + k_{2M}^2/4}$$

$\bar{k}_{1m} = \min_j\{\bar{k}_{1j}\}$, $k_{2m} = \min_j\{k_{2j}\}$, $k_{2M} = \max_j\{k_{2j}\}$, $k_{1M} = \max_j\{k_{1j}\}$, $\varrho = \max_{x_1\in\mathcal{Q}_1}\|x_1\|$, $\mathcal{Q}_1 = \mathcal{Q}_{11} \cap \mathcal{Q}_{12}$, $\mathcal{Q}_{11} = \{x_1 \in \mathbb{R}^n : |x_{1j}| < \bar{b}_{1j}/\bar{k}_{1j}, j = 1, \ldots, n\}$, $\mathcal{Q}_{12} = \{x_1 \in \mathbb{R}^n : |x_{1j}| \le L_{1j}/k_{1j}, j = 1, \ldots, n\}$, $\mathcal{Q}_2 = \{x_2 \in \mathbb{R}^n : |x_{2j}| \le L_{2j}/k_{2j}, j = 1, \ldots, n\}$,

$$Q_1 = \begin{pmatrix} \bar{k}_{1m} & -\epsilon\mu_M \\ -\epsilon\mu_M & \mu_m \end{pmatrix} \; , \; Q_2 = \begin{pmatrix} k_{1M} + k_g & \epsilon\mu_M \\ \epsilon\mu_M & \mu_M \end{pmatrix}$$

$$Q_3 = \begin{pmatrix} \epsilon\bar{k}_{1m} & -\epsilon k_{2M}/2 \\ -\epsilon k_{2M}/2 & k_{2m} - \epsilon k_C\varrho - \epsilon\mu_M \end{pmatrix}$$

we have, on $\mathcal{Q}_1 \times \mathcal{Q}_2$, that

$$c_1\|x\|^2 \le V_2(x) \le c_2\|x\|^2$$

$$\dot{V}_2(x) \le -c_3\|x\|^2$$

with $c_1 = \lambda_m(Q_1)/2 > 0$, $c_2 = \lambda_M(Q_2)/2 > 0$ and $c_3 = \lambda_m(Q_3) > 0$, whence we conclude —by (Khalil, 2002, Theorem 4.10)— that the origin $(x_1, x_2) = (0_n, 0_n)$ is a (locally) exponentially stable equilibrium of the closed-loop system. The details omitted in this sketch of the proof of Corollary 3.1 will be thoroughly developed in future communications with more relaxed space limitations.

4. SIMULATION RESULTS

We implemented the proposed control scheme through numerical simulations considering the 2-DOF revolute-joint robot manipulator model used in (Zavala-Río and Zamora-Gómez, 2017), characterized by

$$H(q) = \begin{pmatrix} 2.351 + 10.168\cos q_2 & 0.102 + 0.084\cos q_2 \\ 0.102 + 0.084\cos q_2 & 0.102 \end{pmatrix}$$

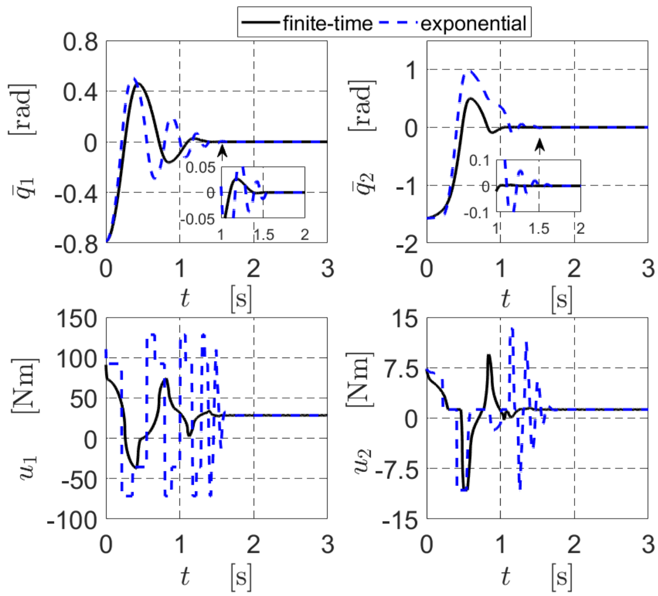

Fig. 1. Finite-time *vs* exponential stabilization

$$C(q,\dot{q}) = \begin{pmatrix} -0.084\dot{q}_2 \sin q_2 & -0.084(\dot{q}_1+\dot{q}_2)\sin q_2 \\ 0.084\dot{q}_1 \sin q_2 & 0 \end{pmatrix}$$

$$g(q) = \begin{pmatrix} 38.465 \sin q_1 + 1.825 \sin(q_1+q_2) \\ 1.825 \sin(q_1+q_2) \end{pmatrix}$$

Assumptions 2.1–2.3 are thus satisfied (this is a direct consequence of the revolute nature of both joints of the considered manipulator); in particular Assumption 2.3 is fulfilled with $B_{g1} = 40.29$ Nm, $B_{g2} = 1.825$ Nm and $k_g = 40.37$ Nm/rad. Input saturation bounds are $T_1 = 150$ Nm and $T_2 = 15$ Nm, whence Assumption 2.4 is corroborated to be fulfilled with $\eta = 3$. For the sake of simplicity, units will be subsequently omitted.

The proposed design methodology was applied under the consideration of the following function definitions

$$\sigma_u(\varsigma;\beta,a) = \text{sign}(\varsigma)\max\{|\varsigma|^{\beta}, a|\varsigma|\} \quad (24a)$$

$$\sigma_b(\varsigma;\beta,a,M) = \text{sign}(\varsigma)\min\{|\sigma_u(\varsigma;\beta,a)|, M\} \quad (24b)$$

for constants $\beta > 0$, $a \in \{0,1\}$, and $M > 0$. Examples are shown in (Zavala-Río and Zamora-Gómez, 2017, §5).

We present a simulation test focusing on the comparison among the two types of convergence: finite-time *vs* exponential. The implementation was run taking the desired configuration at $q_d = \begin{pmatrix}\pi/4 & \pi/2\end{pmatrix}^T$ [rad] and initial conditions as $q(0) = \dot{q}(0) = 0_2$. Based on the functions in Eqs. (24), we define, for every $j = 1,2$,

$$\sigma_{ij}(\varsigma) = \sigma_b(\varsigma;\beta_i,a_{ij},M_{ij}) \qquad i = 1,2 \quad (25)$$

with $a_{ij} = 0$, $i = 1,2$, $j = 1,2$. Conditions on their parameters under which (9) is fulfilled are:

$$k_{1j} > k_g(2B_{gj})^{(1-\beta_1)/\beta_1} \quad (26a)$$

$$M_{1j} > 2B_{gj} \quad (26b)$$

Let us note, from the involved functions, as defined through Eq. (25), that $B_j = M_{1j} + M_{2j}$, $j = 1,2$ (see (8)). Thus, by fixing $M_{11} = 82$, $M_{21} = 18$, and $M_{12} = M_{22} = 6$, (8) and (26b) are simultaneously satisfied. The control gain values were chosen taking care that inequality (26a) was satisfied.

Figure 1 shows results obtained taking $\beta_1 = 1/2$ and $\beta_2 = 2/3$, and the control gains were taken, for both (finite-time and exponential) controllers, as: $K_1 = \text{diag}[5000, 200]$ and $K_2 = \text{diag}[150, 5]$. One sees that the proposed scheme achieves both types of convergence avoiding input saturation, with the closed-loop trajectory arising through the exponential controller presenting a longer and more important transient. On the other hand, the finite-time stabilizer shows a more efficient ability to counteract the inertial effects through control signals with considerably less and lower variations during the transient.

5. CONCLUSIONS

Global continuous control of robot manipulators with input constraints guaranteeing finite-time or exponential stabilization has been made possible and further simplified through desired gravity compensation. This controller is not a simple extension of the on-line compensation case but it has rather proven to need more involved requirements resulting from a closed-loop analysis with considerably higher degree of complexity. Simulation results have shown the actual ability of the proposed approach to guarantee the considered types of convergence avoiding input saturation, with finite-time control signals giving rise to less and lower variations during the transient. A more detailed implementation test study focusing on further aspects on the closed-loop performance is intended to be presented on future communications with more relaxed space restrictions.

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