Nonlinear passive control of a class of coupled partial differential equation models

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Abstract: In this work, the stabilization problem of a possible open-loop unstable steady-state for a class of semilinear parabolic partial differential equation models with an averaged measurement and homogeneously distributed control action is addressed. Following notions of passivity-based control for finite-dimensional systems, a feedback passive control is constructed. The combination of Lyapunov and modal techniques gives sufficient conditions to ensure the stability of the closed-loop system by characterizing the zero dynamics behavior in terms of the sensor location and the controller gain. For implementation purposes, an estimator with a pointwise innovation scheme is considered. The performance of the designed controller is shown by numerical simulations.

Keywords: distributed parameter systems, passivity based control, sensor and actuator placement

1. INTRODUCTION

Passivity based control for nonlinear finite-dimensional systems has proven to be a useful tool for control synthesis, it has been applied to stable and unstable plants, from mechanical and electrical systems, see, e.g., [Ortega et al., 2013], to chemical processes, see, e.g., [Doerfler et al., 2009, Sira-Ramirez and Angulo-Nunez, 1997]. The extension to infinite dimensional systems has been done following the early- and late-lumping approaches. In the early-lumping framework, the partial differential equation (PDE) model is approximated using a finite-dimensional model and then existing results on passivity based control has been applied. On the other hand, in the late-lumping approach the extension of passivity concepts has been performed for some types of PDE models exploiting its distributed structure and considering different input and output configurations.

In the context of early-lumping approaches, in [Franco-de los Reyes and Álvarez, 2017, Nájera et al., 2015] finite differences are used to obtain a finite-dimensional model of a tubular reactor and then a feedback passive controller is designed. In [Christofides, 2012], Galerkin and approximated inertial manifolds methods are used to approximate parabolic PDE models and then geometric control tools are used. Regarding late-lumping techniques, in [Christofides, 2012] geometric control is used for the stabilization of a plug flow tubular reactor with collocated sensor and actuator setup. Passivity for linear PDE models with collocated setup is analyzed in [Bondarko and Fradkov, 2002], while semilinear parabolic systems are considered in [Wang and Wu, 2014] where a feedback passivity-based controller is built exploiting Lyapunov theory. In the framework of thermodynamics in [Alonso et al., 2000, Ruszkowski et al., 2005] passivity-based control is studied for semilinear PDEs models. In the more general context of dissipativity-based control design, in [Schaum and Meurer, 2019] stabilization through linear output-feedback control of a semilinear heat equation with collocated sensor-actuator setup and an output dependent nonlinearity is considered.

In the present work, the ideas exploited in [Franco-de los Reyes and Álvarez, 2017, Nájera et al., 2015] following the early-lumping approach to design a passive controller for a tubular reactor model are extended to a class of infinite-dimensional systems. A passive controller is introduced for a class of diffusion-convection-reaction semilinear systems with regional sensor and homogeneous actuator. Conditions for the stability of the closed-loop system are established based on the stability of the origin of the related zero dynamics, which can be analyzed as a Lur’e system, i.e., an interconnection of a linear dynamic...
Consider the diffusion-convection-reaction system with real eigenvalues $\lambda_n$, $n \in \mathbb{N}$ fulfilling $0 > \lambda_1 \geq \lambda_2 \geq ...$ for which the algebraic and geometric multiplicities are the same, and whose eigenfunctions $\phi_n$, given by the solution to the eigenvalue problem

$$\mathcal{A}_i \phi_n(z) - \lambda_n \psi_n(z) = 0, \quad n \in \mathbb{N},$$

form a Riesz basis, i.e., $\langle \psi_n, \psi_k \rangle = \delta_{n,k},$ where $\delta_{n,k}$ is the Kronecker delta.

(A2) The operator $\mathcal{A}_2$ generates a $C_0$-semigroup of contractions $S_2(t) = e^{\mathcal{A}_2 t}$ which satisfy $\|S_2(t)\| \leq e^{-\nu_2 t}$ where $\nu_2$ is its growth bound [Curtain and Zwart, 2012].

(A3) The source term $\varphi$ belongs to the sector $[-\kappa, \kappa]$ and satisfies $\|\varphi(z,x_1)\| \leq \kappa \|x_1\| \forall x_1 \in \mathcal{H}$ uniformly in $z$ and $\varphi(z, 0) = 0$.

The control problem consists in selecting the sensor location $\zeta$ and design an output feedback controller such that the corresponding closed-loop system has the zero profile as unique and exponentially stable steady-state.

3. FEEDBACK-PASSIVE CONTROL

Here, for system (3) a passive controller is constructed, the closed-loop stability is ensured by means of the stability of the zero dynamics and its dependency on the sensor location.

3.1 Controller construction

Consider the output (3c) and take its first time derivative

$$\dot{y} = \frac{\partial}{\partial t} Cx_1 = C\dot{x}_1 = C(A_1 x_1 + \varphi(x_1)) + CBu.$$

If $CB \neq 0$ then the characteristic order from $y$ to $u$ is equal to one [Christofides, 2012]. Note that this concept is the extension of the relative degree property for finite-dimensional systems. Taking into account the considered input and output operators it follows that

$$CB = \langle \gamma, \beta \rangle = \frac{1}{\gamma} \neq 0, \quad \forall z \in (0, 1), \quad \forall t \in \mathbb{R}_+.$$

Consequently the characteristic index is always one and the existence of the state-feedback controller

$$u = 2\epsilon \left[ v - C (A_1 x_1 + \varphi(x_1)) \right],$$

system and an static nonlinearity and by applying modal analysis. Stability conditions are established in terms of the sensor location and the controller gain. In comparison to [Franco-de los Reyes et al., 2019a,b], where a similar approach has been followed, here different system structure, controller and sensor-actuator setup are considered.

The rest of the paper is organized as follows, in Section 2 the control problem is introduced, in Section 3 the passive controller is built and closed-loop stability ensured. In Section 4 the output-feedback version of the proposed controller is presented. Simulation results are shown in Section 5 and final conclusions are given in Section 6.
is ensured and it can be used to stabilize the (possibly open-loop unstable) zero profile of system (3), as it will be established later.

**Remark 1.** Using $S(x) = \frac{1}{2}y^2 \geq 0$ as storage function it follows that $\frac{dS(x)}{dt} = y v$, which shows the passivity property introduced by the controller (6). In particular, using $v = -ky$ the exponentially stable output dynamics $\dot{y} = -ky$, $y(0) = y_0$ is enforced.

The corresponding closed-loop dynamics are given by

\[
\begin{align*}
\dot{x}_1 &= A_1^c x_1 + \Delta \varphi(x_1) - 2cB \dot{y}, \quad x_1(0) = x_{10}, \quad \text{(7a)} \\
\dot{x}_2 &= A_2 x_2 + a_{21} x_1, \quad x_2(0) = x_{20}, \quad \text{(7b)} \\
y &= C x_1
\end{align*}
\]

where $A_1^c$, with domain $\mathcal{D}(A_1^c) = \mathcal{D}(A_1)$, is the closed-loop operator and $\varphi$ a modified nonlinear term, which are defined as

\[
\begin{align*}
A_1^c x_1 &= A_1 x_1 - 2cB C A_1 x_1, \quad \text{(7d)} \\
\Delta \varphi(x_1) &= \varphi(x_1) - 2cB \varphi(x_1), \quad \text{(7e)}
\end{align*}
\]

Note that in (7c) the nonlinear component of the controller (6) takes the weighted (by $2B$) and averaged (by $C$) nonlinear term $\varphi$ and subtract it from the original nonlinearity, potentially mitigating its destabilizing effect. By assumption (A3) it follows that

$$||\Delta \varphi(x_1)|| \leq \kappa_0 ||x_1||,$$

where $\kappa_1(\zeta)$ is sensor location dependent highlighting that $\zeta$ is a key design degree of freedom in the stabilization task. For the stability assessment the characterization of the zero dynamics is addressed next.

### 3.2 Zero dynamics

The related zero dynamics associated to (7) are given by

\[
\begin{align*}
\dot{x}_1^z &= A_1^z x_1 + \Delta \varphi(x_1^z), \quad x_1^z(0) = x_{10}^z, \quad \text{(8a)} \\
\dot{x}_2^z &= A_2 x_2 + a_{21} x_1^z, \quad x_2^z(0) = x_{20}^z, \quad \text{(8b)} \\
y &= C x_1^z = 0,
\end{align*}
\]

with the zero dynamics operator $A_1^z$ defined as

$$A_1^z x_1 = A_1^c x_1, \quad \mathcal{D}(A_1^z) = \{x \in \mathcal{D}(A_1) | C x = 0\}. \quad \text{(8d)}$$

In the following Lemma sufficient conditions for the stability of the solution $(x_1^z, x_2^z) = (0, 0)$ are established.

**Lemma 1.** Let assumptions (A1)-(A3) hold and additionally assume that the origin is the unique steady-state for the zero dynamics. If the operator $A_1^z$ defined in (8d) generates a $C_0$-semigroup of contractions $S_1^z(t) = e^{A_1^z t}$ with growth bound $\nu_2$ which satisfies

$$\nu_2 - \kappa_0 := \nu_2 > 0,$$

then $(x_1^z, x_2^z) = (0, 0)$ is globally exponentially stable in the $L^2$-norm and input-to-state stable with respect to additive disturbances.

**Proof:** Considering a bounded additive disturbance $z(t) \in L^2$ in the $x_1^z$ dynamics (8a), the formal solutions for (8) are given by

\[
\begin{align*}
x_1^z(t) &= x_{10}^z + \int_0^t S_1^z(t - r) [\Delta \varphi(x_1^z(r)) + \zeta(\tau)] \, dr, \\
x_2^z(t) &= x_{20} + \int_0^t a_{21} S_1^z(t - r) x_1^z(r) \, dr,
\end{align*}
\]

taking norms, using Assumption (3) and applying the triangle inequality the following is obtained

$$||x_1^z(t)|| \leq ||x_{10}^z|| e^{-\nu_2 t} + \int_0^t e^{-\nu_2 (t - r)} (\kappa_0 ||x_1^z|| + ||\zeta(\tau)||) \, dr,$$

$$||x_2^z(t)|| \leq ||x_{20}|| e^{-\nu_2 t} + ||a_{21}|| \int_0^t e^{-\nu_2 (t - r)} ||x_1^z|| \, dr.$$ Determine the right-hand-side of the above inequalities as $\kappa_1$, $i = 1, 2$. It holds that $\kappa_1(t) = ||x_i||, i = 1, 2,$

$$||x_1^z(t)|| < \kappa_1(t), ||x_2^z(t)|| < \kappa_1(t), \quad \text{for } t \geq 0 \text{ and}
\]

\[
\begin{align*}
\frac{d}{dt} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} &\leq -\nu_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} ||z||.
\end{align*}
\]

Due to the triangular matrix in the above and that by Assumption (A2) $\nu_2 > 0$, the following holds: (i) $z(t) = 0$ implies that the zero solution $x_i = 0, i = 1, 2$, is exponentially stable if (9) is fulfilled and as a consequence the zero solution of the zero dynamics is exponentially stable in the $L^2$-norm, and (ii) when the disturbance $\zeta(t)$ is present the following is satisfied

$$||x_1^z(t)|| < \kappa_1(t), ||x_2^z(t)|| < \kappa_1(t).$$

where $\cdot||$ denotes the supremum-norm. This implies input-to-state stability [Karafyllis and Krstic, 2019, Sonntag, 1995].

### 3.3 Closed-loop stability

The closed-loop dynamics (7) can be rewritten as the following cascaded interconnection

\[
\begin{align*}
y &= -ky, \quad y(0) = y_0, \quad \text{(10a)} \\
\dot{x}_1 &= A_1^c x_1 + \Delta \varphi(x_1) - 2cB \dot{y}, \quad x_1(0) = x_{10}, \quad \text{(10b)} \\
\dot{x}_2 &= A_2 x_2 + a_{21} x_1, \quad x_2(0) = x_{20}. \quad \text{(10c)}
\end{align*}
\]

Since the domain of the closed-loop dynamics differs from the one of the zero dynamics, write

$$x_1 = x_1^z + x_1^+, \quad x_2 = x_2^+,$$

Consequently, it holds that the output $y$ converge to zero together with $\dot{x}_1$ and there exists a positive constant $M$ such that

$$|y| \leq |y_0| e^{-kt} \Rightarrow ||\dot{x}_1|| \leq M||x_1|| e^{-kt}.$$ Considering the formal solutions of the closed-loop system with $||x_1|| \leq ||x_1^z|| + ||x_1^+||$, applying the triangle inequality and using the input-to-state stability property of the zero dynamics with $\zeta(t) = -2cB \dot{y}$, the following holds

$$||x_1|| \leq ||x_1^z|| + \eta |y_0| e^{-\nu_2 t} + (M||x_1^z|| + |\eta|y_0) e^{-kt}$$

$$||x_2|| \leq ||x_{20}|| e^{-\nu_2 t} + \frac{\eta}{\nu_2} ||x_1||,$$

where $\eta = \frac{2e|\bar{B}|k}{\nu_2 - k}$. Consequently $(x_1^z, x_2^z)$ converge exponentially to zero, in the $L^2$-norm, with rate min $\{\nu_2, k\}$. This result is stated in the following proposition.

**Proposition 1.** Let the assumptions of Lemma 1 hold. Then, the controller (6) exponentially stabilizes the origin of the system (3) in the $L^2$-norm with convergence rate given by min $\{\nu_2, k\}$. 
3.4 Sensor placement

According to Proposition 1 and Lemma 1, the key property for the functioning of the proposed control scheme is the exponential stability of the origin of the zero dynamics in the \( L^2 \)-norm. In the following this property is characterized in terms of the sensor location using Lyapunov techniques and modal representation. For this aim, consider the following Lyapunov functional

\[
V = \frac{1}{2}(x_1^2, x_2^2) > 0.
\]

Its time derivative along the trajectories of (8a) reads

\[
\frac{dV}{dt} = \frac{1}{2}\langle \mathcal{A}_1 x_1^2 + \Delta \varphi(x_1^2), x_1^2 \rangle + \frac{1}{2}\langle x_1, \partial_1 \varphi(x_1) \rangle
\]

\[
= \frac{1}{2}(\mathcal{A}_1 x_1^2 + \Delta \varphi(x_1^2), x_1^2) + \frac{1}{2}(x_1, \mathcal{A}_1 x_1^2 + \Delta \varphi(x_1^2))
\]

\[
= \langle \mathcal{A}_1 x_1^2, x_1^2 \rangle - 2\epsilon \mathcal{C}_1 \mathcal{A}_1 \mathcal{B}_1 + (\Delta \varphi(x_1^2), x_1^2)
\]

In the following the notation \( \sum = \sum_{n=1}^{\infty} \) and \( \sum' = \sum_{n=2}^{\infty} \) is used. From Assumption (A1), the first state has the modal representation \( x_1 = \sum \alpha_n \psi_n \) and the output can be written as \( y = \sum \alpha_n \psi_n \) where \( \alpha_n = \langle \psi_n, \psi_n \rangle \) and \( \psi_n = (\gamma_n, \psi_n) \). Furthermore, the action of the operator \( \mathcal{A}_1 \) can be expressed as \( \mathcal{A}_1 x_1 = \sum \lambda_n \alpha_n \psi_n \), thus it follows that \( \mathcal{C}_1 x_1^2 = \sum \lambda_n \alpha_n \psi_n \) and \( (\mathcal{B}, x_1) = \sum \alpha_n \psi_n \) where \( \psi_n = (\beta_n, \psi_n) \). Considering all the introduced modal representations, the time derivative of \( V \) reads

\[
\frac{dV}{dt} = \sum \lambda_n \alpha_n (\psi_n, \sum \alpha_n \psi_n) - 2\epsilon \sum \lambda_n \alpha_n \psi_n \sum \alpha_n \beta_j
\]

\[
+ (\Delta \varphi(x_1^2), x_1^2),
\]

\[
= \sum \lambda_n (\sum \alpha_n \sum \alpha_n \beta_j + \sum \lambda_n \alpha_n \psi_n + \sum \lambda_n \alpha_n \psi_n)
\]

\[
= \lambda_1 \alpha_1^2 + \sum \lambda_n \alpha_n \psi_n + \sum \lambda_n \alpha_n \psi_n
\]

\[
- 2\epsilon \lambda_1 \alpha_1 \psi_n + \sum \lambda_n \alpha_n \psi_n.
\]

Since \( y = 0 \) it holds that \( \alpha_1 = -\frac{1}{\lambda_1} \sum \alpha_n \psi_n \) and thus

\[
\frac{dV}{dt} = \sum \sum' \frac{\lambda_n}{\lambda_1} \alpha_n \psi_n + \frac{\lambda_n}{\lambda_1} \alpha_n \psi_n
\]

\[
- 2\epsilon \sum \sum \lambda_n \alpha_n \psi_n \sum \alpha_n \beta_j
\]

\[
+ (\Delta \varphi(x_1^2), x_1^2).
\]

Writing the summations above in a quadratic form yields

\[
\frac{dV}{dt} = a^T Z a + \langle \Delta \varphi(x_1^2), x_1^2 \rangle
\]

where \( a = [\alpha_n]_{n=2}^{\infty} \) is an infinite-dimensional vector and \( Z = [z_{n,j}] \), \( n,j = 2, \ldots \) is an infinite-dimensional matrix with entries defined as follows

\[
z_{n,j} = \left\{ \begin{array}{ll}
\lambda_n + \frac{\lambda_n}{\lambda_1} c_n^2 & n = j \\
\frac{\lambda_n}{\lambda_1} c_n c_j - 2\epsilon \sum \lambda_n c_n \sum \alpha_n \beta_j & n \neq j
\end{array} \right.
\]

Thus it holds that

\[
\frac{dV}{dt} \leq -\nu_z \| x_2 \|^2 + \langle \tilde{\epsilon} \| x_1 \|, x_1 \rangle \leq -\nu_z \| x_2 \|^2,
\]

\[
\leq (\nu_z - \tilde{\epsilon}) V(x_1),
\]

where \( \nu_z = \sup_{n \geq 2} \lambda(Z) \). To ensure that the time derivative of \( V(x_2^2) \) is negative definite, the matrix \( Z \) must be Hurwitz. This can be established by applying the Gersgorin theorem for infinite-dimensional matrices (see Theorem (16c) in Aleksić et al. [2014]). Since \( \rho \) and \( \phi \) are the eigenvalues of the system, note that two of them are stable and the one in the middle, denoted as \( \tilde{x}_1^*, \tilde{x}_2^* \), is unstable and selected for closed-loop operation. The open-loop dynamic response of the system initialized near the unstable profile pair is shown in Fig. 1 (right) and confirms its instability.
Fig. 1. Open-loop system steady-states (left) and profile evolution (instability of the target steady-state).

Fig. 2. Steady-states dependency on the sensor location: triplicity for \( \zeta \in [0, 0.87] \), uniqueness for \( \zeta \in [0.87, 1] \).

The eigenvalues and eigenfunctions are given by

\[
\lambda_n = \left( \frac{1}{2\zeta} \right)^2 - \omega_n^2, \quad \tan(\omega_n) = \frac{2\nu}{\omega_n^2 - \frac{1}{4\zeta^2}}, \quad \omega_n \neq 0,
\]

\[
\psi_n = B_n (2d\omega_n \cos(\omega_n z) + \sin(\omega_n z)),
\]

where \( B_n \) are normalization constants. The series \( b_n \) and \( c_n \) are given by

\[
b_n = \frac{2B_n}{\omega_n^2 + \left( \frac{1}{2\zeta} \right)^2} \left( \omega_n + e^{-\frac{1}{2\zeta}} \left( d \left( \omega_n^2 - \frac{1}{2\zeta} \right) \sin(\omega_n) + \omega_n \cos(\omega_n) \right) \right)
\]

\[
c_n = B_n e^{\frac{1}{2\zeta}} \left( 2d\omega_n (\cos(\omega_n \zeta) + \sin(\omega_n \zeta)) \right).
\]

Note that \( b_n \) are constants, and that \( c_n \) depends on the sensor location \( \zeta \). According to the previous sections, \( \zeta \) must be selected to satisfy the conditions of Lemma 1 and Lemma 2. The first assumption is the uniqueness of the steady-state solution for the zero dynamics, i.e., the uniqueness of the zero profile pair for the boundary value problem

\[
0 = A_1^t x_1 + \varphi(x_1^t), \quad x_1 \in D(A_1^t)
\]

\[
0 = A_2 x_2 + a_21 x_1^t, \quad x_2 \in D(A_2)
\]

\[
0 = C x_1 = y,
\]

where the restriction \( y = x_1^t(\zeta) = 0 \) for \( \zeta \in [0, 1] \) is the unique degree of freedom. The uniqueness of the solution \( (x_1^t, x_2) = 0 \) requires that for the first state \( x_1 = 0 \) must be a unique solution. The analysis of the above boundary value problem is done by constructing a bifurcation diagram (based on a finite differences approximation) using the Matcont software package [Dhooge et al., 2003]. The obtained result is shown in Fig. 2. It can be seen that the zero solution is unique if \( \zeta \in I_\pm = (0.87, 1) \). Accordingly \( \zeta = 0.9 \) is selected so that the conditions of Lemma 2 are fulfilled, i.e., \( c_1(\zeta) \neq 0 \) and \( \nu_z \approx -4.13 \). The numerical computation of \( \kappa \) can be done using a Lipschitz constant that for this case gives \( \kappa = 3 \). Thus condition (9) is fulfilled with \( \nu_z = -1.13 \). This ensures the convergence to zero of the zero dynamics (that determines the rate of convergence of the closed-loop states) and for the closed-loop system (15). The gain \( k \) is used to accelerate the rate of convergence of the output.

In Fig. 3 the closed-loop behavior with controllers (6) and (14) with \( k = 3 \) is shown in original coordinates. The system is initialized at the lower steady-state. It can be seen that for the state feedback case the output goes to zero in about 1 time unit so the system in the zero dynamics converge to the desired steady-state in 2 time units. For the output-feedback case the convergence is slower due to the estimation convergence time. In Fig. 4 the approximated \( L^2 \)-norms of the corresponding state
profiles are shown. It can be seen that the zero dynamics response is the best attainable behavior.

All simulation were carried out using finite differences with 100 collocation points, approximating the integral with the trapezoidal rule using the Matlab command trapez and solving the obtained system of ordinary differential equations with the method ode15s in Matlab.

6. CONCLUSIONS

The output-feedback stabilization problem of an unstable profile of a class of semilinear PDE models with averaged measurement and homogeneous control action has been addressed. The controller scheme is constructed following a similar notion to feedback passivity for nonlinear finite-dimensional systems. The close-loop stability is ensured by the characterization of the zero dynamics in terms of the sensor location by using a combined approach with Lyapunov and modal techniques. For implementation purposes an observer scheme is added to the design. Numerical simulations show the satisfactory performance of the proposed approach.

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